ASUPRA UNEI FORMULE DE TIP TAYLOR PENTRU FUNCȚIILE DE TREI VARIABILE

de

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În această lucrare ne vom ocupa de extinderea unei formule de tip Taylor, studiată în [1] pentru funcțiile de două variabile, la funcțiile de trei variabile. Scopul acestei lucrări va consta în stabilirea unei expresii integrale simple pentru restul formulei pe care o vom studia.

1) Să considerăm polinomul

$$P(x,y,z) = \sum_{i=0}^{l} \sum_{j=0}^{m} \sum_{k=0}^{n} A_{i,j,k} x^i y^j z^k \tag{1}$$

de grad $(l,\ m,\ n)$.

Derivata parțială de ordinul $(\alpha,\ \beta,\ \gamma)$, $(0 \leqq \alpha \leqq l,\ 0 \leqq \beta \leqq m,\ 0 \leqq \leqq \gamma \leqq n)$, a acestui polinom este dată de formula

$$\frac{\partial^{\alpha+\beta+\gamma} P(x,\ y,\ z)}{\partial x^\alpha \partial y^\beta \partial z^\gamma} = \sum_{i=0}^{l} \sum_{j=0}^{m} \sum_{k=0}^{n} A_{i,j,k} L_{i\alpha} M_{j\beta} N_{k\gamma} x^{i-\alpha} y^{j-\beta} z^{k-\gamma} \tag{2}$$

unde

$$L_{i\alpha} = i(i-1)\ldots(i-\alpha+1)\ ; \quad M_{j\beta} = j(j-1)\ldots(j-\beta+1)\ ;$$

$$N_{k\gamma} = k(k-1)\ \ldots\ (k-\gamma+1).$$

Din (2), făcînd $x = y = z = 0$, se pot deduce următoarele expresii pentru coeficienții polinomului

$$A_{i,j,k} = \frac{1}{i!\,j!\,k!} \left. \frac{\partial^{i+j+k} P(x,\ y,\ z)}{\partial x^i \partial y^j \partial z^k} \right|_{\substack{x=0\\ y=0\\ z=0}} = \frac{1}{i!\,j!\,k!} P^{(i+j+k)}_{x^i y^j z^k}\ (0,\ 0,\ 0).$$

Rezultă că polinomul (1) se scrie şi sub forma

$$P(x, y, z)=\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} P_{x^iy^jz^k}^{(i+j+k)}(0, 0, 0)\frac{x^iy^jz^k}{i!j!k!}. \tag{3}$$

Folosind rezultatele de mai sus, se poate deduce pentru $P(x, y, z)$ următoarea dezvoltare

$$P(x,y,z)=\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} B_{i,j,k}(x-a)^i(y-b)^j(z-c)^k \tag{4}$$

unde

$$B_{i,j,k}=\frac{1}{i!j!k!}P_{x^iy^jz^k}^{(i+j+k)}(a, b, c) \tag{5}$$

2. Să considerăm o funcţie $f(x, y, z)$ care într-o vecinătate oarecare $V(M)$ a punctului $M(a, b, c)$ admite derivatele parţiale următoare

$$\frac{\partial^{\alpha+\beta+\gamma}f(x, y, z)}{\partial x^\alpha\partial y^\beta\partial z^\gamma}\begin{pmatrix}\alpha=\overline{0, l+1}\\ \beta=\overline{0, m+1}\\ \gamma=\overline{0, n+1}\end{pmatrix},$$

despre care presupunem că sînt funcţii continue pe $V(M)$.

Ataşăm funcţiei $f(x, y, z)$ următorul polinom

$$T(x, y, z)=\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} f_{x^iy^jz^k}^{(i+j+k)}(a, b, c)\frac{(x-a)^i(y-b)^j(z-c)^k}{i!j!k!} \tag{6}$$

Să observăm că polinomul (6) şi derivatele sale parţiale, pînă la ordinul (l, m, n) inclusiv, iau în punctul $M(a, b, c)$ aceleaşi valori ca şi funcţia $f(x, y, z)$ şi derivatele sale corespunzătoare, adică avem

$$\left.\frac{\partial^{\alpha+\beta+\gamma}T(x,y,z)}{\partial x^\alpha\partial y^\beta\partial z^\gamma}\right|_{\substack{x=a\\y=b\\z=c}}=\left.\frac{\partial^{\alpha+\beta+\gamma}f(x, y, z)}{\partial x^\alpha\partial y^\beta\partial z^\gamma}\right|_{\substack{x=a\\y=b\\z=c}},\begin{pmatrix}\alpha=\overline{0, l+1}\\ \beta=\overline{0, m+1}\\ \gamma=\overline{0, n+1}\end{pmatrix} \tag{7}$$

Polinomul (6) este un polinom de tip Taylor, care reprezintă funcţia $f(x, y, z)$ cu aproximaţie. Pentru studiul acestei aproximaţii trebuie să studiem diferenţa

$$R(x, y, z)=f(x, y, z)-T(x, y, z). \tag{8}$$

Din (7) şi (8) se deduce că

$$\left.\frac{\partial^{\alpha+\beta+\gamma}R(x, y, z)}{\partial x^\alpha\partial y^\beta\partial z^\gamma}\right|_{\substack{x=a\\y=b\\z=c}}=0\begin{pmatrix}\alpha=\overline{0, l+1}\\ \beta=\overline{0, m+1}\\ \gamma=\overline{0, n+1}\end{pmatrix} \tag{9}$$

şi că

$$\frac{\partial^{l+1}R(x,y,z)}{\partial x^{l+1}}=\frac{\partial^{l+1}f(x,y,z)}{\partial x^{l+1}},\quad \frac{\partial^{m+1}R(x,y,z)}{\partial y^{m+1}}=\frac{\partial^{m+1}f(x,y,z)}{\partial y^{m+1}},\quad \frac{\partial^{n+1}R(x,y,z)}{\partial z^{n+1}}=\frac{\partial^{n+1}f(x,y,z)}{\partial z^{n+1}},$$

$$\begin{gathered}\frac{\partial^{l+m+2}R(x,y,z)}{\partial x^{l+1}\partial y^{m+1}}=\frac{\partial^{l+m+2}f(x,y,z)}{\partial x^{l+1}\partial y^{m+1}},\\ \frac{\partial^{l+n+2}R(x,y,z)}{\partial x^{l+1}\partial z^{n+1}}=\frac{\partial^{l+n+2}f(x,y,z)}{\partial x^{l+1}\partial z^{n+1}},\\ \frac{\partial^{m+n+2}R(x,y,z)}{\partial y^{m+1}\partial z^{n+1}}=\frac{\partial^{m+n+2}f(x,y,z)}{\partial y^{m+1}\partial z^{n+1}},\\ \frac{\partial^{l+m+n+3}R(x,y,z)}{\partial x^{l+1}\partial y^{m+1}\partial z^{n+1}}=\frac{\partial^{l+m+n+3}f(x,y,z)}{\partial x^{l+1}\partial y^{m+1}\partial z^{n+1}}.\end{gathered}\tag{10}$$

Să presupunem că cubul Δ definit de inegalităţile $a\leq t_1\leq x$, $b\leq t_2\leq y$, $c\leq t_3\leq z$ e conţinut în $V(M)$. Avem lema:

L e m a 1. *Termenul complimentar al formulei de tip Taylor* $f(x, y, z) = (Tx, y, z) + T(x, y, z)$ (11) *se mai poate scrie sub forma* $R(x, y, z) =$

$$= \int_a^x \frac{\partial R(t_1, y,z)}{\partial t_1} dt_1 + \int_b^y \frac{\partial R(x,t_2,z)}{\partial t_2} dt_2 + \int_c^z \frac{\partial R(x,y,t_3)}{\partial t_3} dt_3 - \int_a^x\int_b^y \frac{\partial^2 R(t_1,t_2,z)}{\partial t_1\partial t_2} dt_1 dt_2 -$$

$$- \int_a^x\int_c^z \frac{\partial^2 R(t_1,y,t_3)}{\partial t_1\partial t_3} dt_3 - \int_b^y\int_c^z \frac{\partial^2 R(x,t_2,t_3)}{\partial t_2\partial t_3} dt_2 dt_3 + \int_a^x\int_b^y\int_c^z \frac{\partial^3 R(t_1,t_2,t_3)}{\partial t_1\partial t_2\partial t_3} dt_1 dt_2 dt_3. \tag{12}$$

Verificarea egalităţii (11) se face cu multă uşurinţă.

Dacă introducem notaţia

$$R_1(x, y, z) = \int_a^x \frac{\partial R(t_1,y,z)}{\partial t_1} dt_1 - \int_a^x\int_b^y \frac{\partial^2 R(t_1,t_2,z)}{\partial t_1\partial t_2} dt_1 dt_2 - \int_a^x\int_c^z \frac{\partial^2 R(t_1,y,t_3)}{\partial t_1\partial t_3} dt_1 dt_2 +$$

$$+ \int_a^x\int_b^y\int_c^z \frac{\partial^3 R(t_1,t_2,t_3)}{\partial t_1\partial t_2\partial t_3} dt_1 dt_2 dt_3,$$

restul formulei (11) se scrie

$$\begin{gathered}R(x, y, z) = R_1(x,y,z) + \int_b^y \frac{\partial R(x,t_2,z)}{\partial t_2} dt_2 + \int_c^z \frac{\partial R(x,y,t_3)}{\partial t_3} dt_3 -\\ - \int_b^y\int_c^z \frac{\partial^2 R(x,t_2,t_3)}{\partial t_2\partial t_3} dt_2 dt_3\end{gathered}\tag{13}$$

Să ne ocupăm de $R_1(x, y, z)$. Socotind pe x fix, întrucît $dt_1 = - d(x - t_1)$, vom avea

$$R_1(x,y,z) = -\int_a^x \Bigg[\frac{\partial R(t_1,y,z)}{\partial t_1} - \int_b^y \frac{\partial^2 R(t_1,t_2,z)}{\partial t_1 \partial t_2}\, dt_2 - \int_c^z \frac{\partial^2 R(t_1,y,t_3)}{\partial t_1 \partial t_3}\, dt_3 -$$

$$- \int_b^y \int_c^z \frac{\partial^3 R(t_1,t_2,t_3)}{\partial t_1 \partial t_2 \partial t_3}\, dt_2 dt_3\Bigg] d(x - t_1) = \Bigg\{(x - t_1)\Bigg[\frac{\partial R(t_1,y,z)}{\partial t_1} - \int_b^y \frac{\partial^2 R(t_1,t_2,z)}{\partial t_1 \partial t_2}\, dt_2 -$$

$$- \int_c^z \frac{\partial^2 R(t_1,y,t_3)}{\partial t_1 \partial t_3}\, dt_3 - \int_b^y \int_c^z \frac{\partial^3 R(t_1,t_2,t_3)}{\partial t_1 \partial t_2 \partial t_3}\, dt_2 dt_3\Bigg]\Bigg\}_a^x + \int_a^x (x - t_1)\Bigg[\frac{\partial^2 R(t_1,y,z)}{\partial t_1^2} -$$

$$- \int_b^y \frac{\partial^3 R(t_1,t_2,z)}{\partial t_1^2 \partial t_2}\, dt_2 - \int_c^z \frac{\partial^3 R(t_1,y,t_3)}{\partial t_1^2 \partial t_3}\, dt_3 + \int_b^y \int_c^z \frac{\partial^4 R(t_1,t_2,t_3)}{\partial t_1^2 \partial t_2 \partial t_3}\, dt_2 dt_3\Bigg] dt_1.$$

Partea integrată e nulă, căci

$$\frac{\partial R(t_1,z,y)}{\partial t_1}\bigg|_{t_1=a} - \int_b^y \frac{\partial^2 R(t_1,t_2,z)}{\partial t_1 \partial t_2}\bigg|_{t_1=a} + \int_b^y \int_c^z \frac{\partial^3 R(t_1,t_2,t_3)}{\partial t_1 \partial t_2 \partial t_3}\, dt_2 dt_3\bigg|_{t_1=a} -$$

$$- \int_c^z \frac{\partial^2 R(t_1,y,t_3)}{\partial t_1 \partial t_3}\, dt_3\bigg|_{t_1=a} = \frac{\partial R(t_1,t_2,t_3)}{\partial t_1 \partial t_2 \partial t_3}\bigg|_{\substack{t_1=a\\ t_2=b\\ t_3=c}} = 0,$$

conform formulelor (9).

În continuare avem

$$R(x,y,z) = -\int_a^x (x - t_1)\Bigg[\frac{\partial^2 R(t_1,y,z)}{\partial t_1^2} - \int_b^y \frac{\partial^3 R(t_1,t_2,z)}{\partial t_1^2 \partial t_2}\, dt_2 - \int_c^z \frac{\partial^3 R(t_1,y,t_3)}{\partial t_1^2 \partial t_3}\, dt_3 +$$

$$+ \int_b^y \int_c^z \frac{\partial^4 R(t_1,t_2,t_3)}{\partial t_1^2 \partial t_2 \partial t_3}\, dt_2 dt_3\Bigg] d(x - t_1) = \Bigg\{\frac{(x - t_1)^2}{2!}\Bigg[\frac{\partial^2 R(t_1,y,z)}{\partial t_1^2} - \int_b^y \frac{\partial^3 R(t_1,t_2,z)}{\partial t_1^2 \partial t_2}\, dt_2 -$$

$$- \int_c^z \frac{\partial^3 R(t_1,y,t_3)}{\partial t_1^2 \partial t_3}\, dt_3 + \int_b^y \int_c^z \frac{\partial^4 R(t_1,t_2,t_3)}{\partial t_1^2 \partial t_2 \partial t_3}\, dt_2 dt_3\Bigg]\Bigg\}_a^x + \int_a^x \frac{(x - t_1)^2}{2!}\Bigg[\frac{\partial^3 R(t_1,y,z)}{\partial t_1^3} -$$

$$- \int_b^y \frac{\partial^4 R(t_1,t_2,z)}{\partial t_1^3 \partial t_2}\, dt_2 - \int_c^z \frac{\partial^4 R(t_1,y,t_3)}{\partial t_1^3 \partial t_3}\, dt_3 + \int_b^y \int_c^z \frac{\partial^5 R(t_1,t_2,t_3)}{\partial t_1^3 \partial t_2 \partial t_3}\, dt_2 dt_3\Bigg] dt_1$$

Partea integrată este iarăşi nulă.

Din aproape în aproape se deduce

$$R_1(x, y, z) = \int_a^x \frac{(x-t_1)^l}{l!} \frac{\partial^{l+1} R(t_1, y, z)}{\partial t_1^{l+1}} dt_1 - \int_a^x \int_b^y \frac{(x-t_1)^l}{l!} \frac{\partial^{l+2} R(t_1, t_2, z)}{\partial t_1^{l+1} \partial t_2} dt_1 dt_2 -$$

$$- \int_a^x \int_c^z \frac{(x-t_1)^l}{l!} \frac{\partial^{l+2}(t_1, y, t_3)}{\partial t_1^{l+1} \partial t_3} dt_1 dt_3 + \int_a^x \int_b^y \int_c^z \frac{(x-t_1)^l}{l!} \frac{\partial^{l+3} R(t_1, t_2, t_3)}{\partial t_1^{l+1} \partial t_2 \partial t_3} dt_1 dt_2 dt_3.$$

Înlocuind această expresie a lui $R_1(x, y, z)$ în fomula (13) obținem

$$R(x, y, z) = \int_a^x \frac{(x-t_1)^l}{l!} \frac{\partial^{l+1} R(t_1, y, z)}{\partial t_1^{l+1}} dt_1 + \int_b^y \frac{\partial R(x, y, t_2)}{\partial t_2} dt_2 -$$

$$- \int_a^x \int_c^z \frac{(x-t_1)^l}{l!} \frac{\partial^{l+2}(t_1, y, t_3)}{\partial t_1^{l+1} \partial t_3} dt_1 dt_3 + R_2(x, y, z), \tag{14}$$

unde

$$R_2(x, y, z) = \int_b^y \frac{\partial R(x, t_2, z)}{\partial t_2} dt_2 - \int_b^y \int_c^z \frac{\partial^2 R(x, t_2, t_3)}{\partial t_2 \partial t_3} dt_2 dt_3 -$$

$$- \int_a^x \int_b^y \frac{(x-t_1)^l}{l!} \frac{\partial^{l+2} R(t_1, t_2, z)}{\partial t_1^{l+1} \partial t_2} dt_1 dt_2 + \int_a^x \int_b^y \int_c^z \frac{(x-t_1)^l}{l!} \frac{\partial^{l+3} R(t_1, t_2, t_3)}{\partial t_1^{l+1} \partial t_2 \partial t_3} dt_1 dt_2 dt_3 =$$

$$= \int_b^y \left[\frac{\partial R(x, t_2, z)}{\partial t_2} - \int_c^z \frac{\partial R(x, t_2, t_3)}{\partial t_2 \partial t_3} dt_3 - \int_a^x \frac{(x-t_1)^l}{l!} \frac{\partial^{l+2} R(t_1, t_2, z)}{\partial t_1^{l+1} \partial t_2} dt_1 - \right.$$

$$\left. - \int_a^x \int_c^z \frac{(x-t_1)^l}{l!} \frac{\partial^{l+3} R(t_1, t_2, t_3)}{\partial t_1^{l+1} \partial t_2 \partial t_3} dt_1 dt_3 \right] dt_2.$$

Efectuind calcule analoage cu cele de la $R_1(x, y, z)$, se găsește

$$R_2(x, y, z) = \int_b^y \frac{(y-t_2)^m}{m!} \frac{\partial^{m+1} R(x, t_2, z)}{\partial t_2^{m+1}} dt_2 - \int_b^y \int_c^z \frac{(y-t_2)^m}{m!} \frac{\partial^{m+2}(Rx, t_2, t_3)}{\partial t_2^{m+1} \partial t_3} dt_2 dt_3 -$$

$$- \int_a^x \int_b^y \frac{(x-t_1)^l}{l!} \frac{(y-t_2)^m}{m!} \frac{\partial^{l+m+2} R(t_1, t_2, z)}{\partial t_1^{l+1} \partial t_2^{m+1}} dt_1 dt_2 +$$

$$+ \int_a^x \int_b^y \int_c^z \frac{(x-t_1)^l}{l!} \frac{(y-t_2)^m}{m!} \frac{\partial^{l+m+3} R(t_1, t_2, t_3)}{\partial t_1^{l+1} \partial t_2^{m+1} \partial t_3^{n+1}} dt_1 dt_2 dt_3.$$

Dacă se înlocuieşte această expresie în (14), se obţine

$$R(x,y,z)=\int_a^x\frac{(x-t_1)^l}{l!}\frac{\partial^{l+1}R(t_1,y,z)}{\partial t_1^{l+1}}dt_1+\int_b^y\frac{(y-t_2)^m}{m!}\frac{\partial^{m+1}R(x,t_2,z)}{\partial t_2^{m+1}}dt_2-$$

$$-\int_a^x\int_b^y\frac{(x-t_1)^l}{l!}\frac{(y-t_2)^m}{m!}\frac{\partial^{l+m+2}R(t_1,t_2,z)}{\partial t_1^{l+1}\partial t_2^{m+1}}dt_1dt_2+R_3(x,y,z), \tag{15}$$

unde

$$R_3(x,y,z)=\int_c^z\frac{\partial R(x,y,t_3)}{\partial t_3}dt_3-\int_a^x\int_c^z\frac{(x-t_1)^l}{l!}\frac{\partial^{l+2}(t_1,y,t_3)}{\partial t_1^{l+1}\partial t_3}dt_1dt_3-$$

$$-\int_b^y\int_c^z\frac{(y-t_2)^m}{m!}\frac{\partial^{m+2}R(x,t_2,t_3)}{\partial t^{m+1}\partial t_3}dt_2dt_3+$$

$$+\int_a^x\int_b^y\int_c^z\frac{(x-t_1)^l}{l!}\frac{(y-t_2)^m}{m!}\frac{\partial^{l+m+3}R(t_1,t_2,t_3)}{\partial t_1^{l+1}\partial t_2^{m+1}\partial t_3}dt_1dt_2dt_3=$$

$$=\int_c^z\left[\frac{\partial R(x,y,t_3)}{\partial t_3}-\int_a^x\frac{(x-t_1)^l}{l!}\frac{\partial^{l+2}(t_1,y,t_3)}{\partial t_1^{l+1}\partial t_3}dt_1-\int_b^y\frac{(x-t_2)^m}{m!}\frac{\partial^{m+2}R(x,t_2,t_3)}{\partial t_1^{m+1}\partial t_3}dt_2+\right.$$

$$\left.+\int_a^x\int_b^y\frac{(x-t_1)^l}{l!}\frac{(y-t_2)^m}{m!}\frac{\partial^{l+m+3}R(t_1,t_2,t_3)}{\partial t_2^{l+1}\partial t^{m+1}\partial t_3}dt_1dt_2\right]dt_3$$

Efectuînd o succesiune de calcule (integrări prin părţi) de felul celor care ne-au condus la espresia definitivă a lui $R_1(x,y,z)$, se găseşte

$$R_3(x,y,z)=\int_c^z\frac{(z-t_3)^n}{n!}\frac{\partial^{n+1}R(x,y,t_3)}{\partial t_3^{n+1}}dt_3-$$

$$-\int_a^x\int_c^z\frac{(x-t_1)^l}{l!}\frac{(z-t_3)^n}{n!}\frac{\partial^{l+n+2}R(t_1,y,t_3)}{\partial t_1^{l+1}\partial t_3^{n+1}}dt_1dt_3-$$

$$-\int_b^y\int_c^z\frac{(x-t_2)^m}{m!}\frac{(z-t_3)^n}{n!}\frac{\partial^{m+n+2}R(x,t_2,t_3)}{\partial t_2^{m+1}\partial t_3^{n+1}}dt_2dt_3+$$

$$+\int_a^x\int_b^y\int_c^z\frac{(x-t_1)^l(y-t_2)^m(z-t_3)^n}{l!\,m!\,n!}\frac{\partial^{l+m+n+3}R(t_1,t_2,t_3)}{\partial t_1^{l+1}\partial t_2^{m+1}\partial t_3^{n+1}}dt_1dt_2dt_3.$$

Dacă se înlocuieşte în (15) şi se ţine seama de (10), obţinem expresia integrală a restului formulei de tip Taylor (11)

$$R(x,y,z) = \int_a^x \frac{(x-t_1)^l}{l!} \frac{\partial^{l+1} f(t_1,y,z)}{\partial t_1^{l+1}} dt_1 + \int_b^y \frac{(y-t_2)^m}{m!} \frac{\partial^{m+1} f(x,t_2,z)}{\partial t_2^{m+1}} dt_2 +$$

$$+ \int_c^z \frac{(z-t_3)^n}{n!} \frac{\partial^{n+1} f(x,y,t_3)}{\partial t_3^{n+1}} dt_3 - \int_a^x \int_b^y \frac{(x-t_1)^l}{l!} \frac{(y-t_2)^m}{m!} \frac{\partial^{l+m+2} f(t_1,t_2,z)}{\partial t_1^{l+1} \partial t_2^{m+1}} dt_1 dt_2 -$$

$$- \int_a^x \int_c^z \frac{(x-t_1)^l}{l!} \frac{(z-t_3)^n}{n!} \frac{\partial^{l+n+2} f(t_1,y,t_3)}{\partial t_1^{l+1} \partial t_3^{n+1}} dt_1 dt_3 - \tag{16}$$

$$- \int_b^y \int_c^z \frac{(y-t_2)^m}{m!} \cdot \frac{(z-t^3)^n}{n!} \frac{\partial^{m+n+2} f(x,t_2,t_3)}{\partial t_2^{m+1} \partial t_3^{n+1}} dt_2 dt_3 +$$

$$+ \int_a^x \int_b^y \int_c^z \frac{(x-t_1)^l}{l!} \frac{(y-t_2)^m}{m!} \frac{(z-t_3)^n}{n!} \frac{\partial^{l+m+n+3} f(t_1,t_2,t_3)}{\partial t_1^{l+1} \partial t_2^{m+1} \partial t_3^{n+1}} dt_1 dt_2 dt_3.$$

Avînd în vedere că $(x-t_1)^l$ are semn constant în intervalul (a, x), vom putea scrie

$$R(x,y,z) = \int_a^x \frac{(x-t_1)^l}{l!} \left[\frac{\partial^{l+1} f(t_1,y,z)}{\partial t_1^{l+1}} - \int_b^y \frac{(y-t_2)^m}{m!} \frac{\partial^{l+m+2} f(t_1,t_2,z)}{\partial t_1^{l+1} \partial t_2^{m+1}} dt_2 - \right.$$

$$\left. - \int_c^z \frac{(z-t_3)^n}{n!} \frac{\partial^{l+n+2} f(t_1,y,t_3)}{\partial t_1^{l+1} \partial t_3^{n+1}} dt_3 + \int_b^y \int_c^z \frac{(y-t_2)^m}{m!} \frac{(z-t_3)^n}{n!} \frac{\partial^{l+m+n+3} f(t_1,t_2,t_3)}{\partial t_1^{l+1} \partial t_2^{m+1} \partial t_3^{n+1}} dt_2 dt_3 \right] dt_1 +$$

$$+ \int_b^y \frac{(y-t_2)^m}{m!} \frac{\partial^{m+1} f(x,t_2,z)}{\partial t_2^{m+1}} dt_2 + \int_c^z \frac{(z-t_3)^n}{n!} \frac{\partial^{n+1} f(x,y,t_3)}{\partial t_3^{n+1}} dt_3 -$$

$$- \int_b^y \int_c^z \frac{(y-t_2)^m}{m!} \frac{(z-t_3)^n}{n!} \frac{\partial^{m+n+2} f(x,t_2,t_3)}{\partial t_2^{m+1} \partial t_3^{n+1}} dt_2 dt_3 = \frac{(x-a)^{l+1}}{(l+1)!} \frac{\partial^{l+1} f(\xi,y,z)}{\partial \xi^{l+1}} -$$

$$- \frac{(x-a)^{l+1}}{(l+1)!} \int_b^y \frac{(y-t_2)^m}{m!} \frac{\partial^{l+m+2} f(\xi,t_2,z)}{\partial \xi^{l+1} \partial t_2^{m+1}} dt_2 - \frac{(x-a)^{l+1}}{(l+1)!} \int_c^z \frac{(z-t_3)^n}{n!} \frac{\partial^{l+n+2} f(\xi,y,t_3)}{\partial \xi^{l+1} \partial t_3^{n+1}} dt_3 + \tag{17}$$

$$+ \frac{(x-a)^{l+1}}{(l+1)!} \int_b^y \int_c^z \frac{(y-t_2)^m}{m!} \frac{(z-t_3)^n}{n!} \frac{\partial^{l+m+n+3} f(\xi,t_2,t_3)}{\partial \xi^{l+1} \partial t_2^{m+1} \partial t_3^{n+1}} dt_2 dt_3 +$$

$$+ \int_b^y \frac{(y-t_2)^m}{m!} \frac{\partial^{m+1} f(x,t_2,z)}{\partial t_2^{m+1}} dt_2 + \int_c^z \frac{(z-t_3)^n}{n!} \frac{\partial^{n+1} f(x,y,t_3)}{\partial t_3^{n+1}} dt_3 -$$

$$- \int_b^y \int_c^z \frac{(y-t_2)^m}{m!} \frac{(z-t_3)^n}{n!} \frac{\partial^{m+n+2} f(x,t_2,t_3)}{\partial t_2^{m+1} \partial t_3^{n+1}} dt_2 dt_3,$$

unde $\xi \in (a, x)$.

Dacă ţinem seama că $(y - t_2)^m$ şi $(z - t_3)^n$ au semn constant în intervalul (b, y), respectiv (c, z) vom putea să aplicăm, încă de două ori, prima formulă mediei calculului integral şi în acest mod se ajunge la următoarea formă a restului formulei de tip Taylor (11)

$$R(x,y,z) = \frac{(x-a)^{l+1}}{(l+1)!}\frac{\partial^{l+1}f(\xi,y,z)}{\partial\xi^{l+1}} + \frac{(y-b)^{m+1}}{(m+1)!}\frac{\partial^{m+1}f(x,\eta,z)}{\partial\eta^{m+1}} + \frac{(z-c)^{n+1}}{(n+1)!}\frac{\partial^{n+1}f(x,y,\zeta)}{\partial\zeta^{n+1}} -$$

$$- \frac{(x-a)^{l+1}(y-b)^{m+1}}{(l+1)!(m+1)!}\frac{\partial^{l+m+2}f(\xi,\eta,z)}{\partial\xi^{l+1}\partial\eta^{m+1}} - \frac{(y-b)^{m+1}(z-c)^{n+1}}{(m+1)!(n+1)!}\frac{\partial^{m+n+2}f(x,\eta,\zeta)}{\partial\eta^{m+1}\partial\zeta^{n+1}} -$$

$$- \frac{(x-a)^{l+1}(z-c)^{n+1}}{(l+1)!(n+1)!}\frac{\partial^{l+n+2}f(\xi,y,\zeta)}{\partial\xi^{l+1}\partial\zeta^{n+1}} + \frac{(x-a)^{l+1}(y-b)^{m+1}(z-c)^{n+1}}{(l+1)!(m+1)!(n+1)!}\frac{\partial^{l+m+n+3}f(\xi,\eta,\zeta)}{\partial\xi^{l+1}\partial\eta^{m+1}\partial\zeta^{n+1}}, \tag{18}$$

unde $\xi \in (a, x)$, $\eta \in (b, y)$ şi $\zeta \in (c, z)$ sînt trei numere necunoscute.

Dacă formulele de medie erau aplicate direct la termenii restului (16), atunci în rezultat intrau mai mult de trei numere necunoscute. Ultima formulă generalizează rezultatele obţinute în [1], pentru funcţiile de două variabile, la funcţiile de trei variabile.

Rezultatele de mai sus se pot extinde la funcţiile de mai multe variabile.

Într-o lucrare viitoare ne vom ocupa de cîteva aplicaţii ale rezultatelor din această lucrare.

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SUR UNE FORMULE DU TYPE TAYLOR POUR LES FONCTIONS DE TROIS VARIABLES.

Résumé

Dans le present travail l'auteur determine l'expresion integrale (16) pour le reste de la formule du type Taylor (11). On obtient ainsi la formule (18) pour le reste.