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ON THE CATEGORY OF N-GROUPS

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Summary: We study some general properties of the category of n-groups - Gr_n and a corollary on the intersection of two n-subgroups.

1. Introduction: This paper is an attempt to systematize certain facts concerning objects of the category of n-groups. n-Groups and their homomorphisms are considered not only from the external, categorical, view-point but also, when necessary, from the internal one - which means considering n-groups as sets with a certain structure and their homomorphisms as functions. Thence, the terms homomorphism and morphism will be used interchangeably, depending on the - internal or external - approach.
The symbol $\alpha: A \to B$, where A and B are objects of the same category, usually means a morphism of that category. We will use for categories the same notations as in [2], and for operations in n-groups the notations of [3].

2. The category consisting of n-groups as objects and their homomorphisms as morphisms will be denoted (as in [5] and [6]) by Gr_n. The definition of n-group given by Dörnte in [8] assumed that its carrier was nonempty. In considering the category of n-groups for $n > 2$ it is convenient to admit the empty n-group. The empty n-group is an initial object in Gr_n. The category Gr_n has final objects - the

one-element n-groups. For $n>2$, Gr_n has no zero objects and no zero morphisms.

The class of n-groups is a variety (cf. [4] , [9]); thus in Gr_n monomorphisms and injective homomorphisms coincide.

Using the forgetful functor $\Psi_s: Gr_{k+1} \longrightarrow Gr_{n+1}$ ($n=s.k$) and its left adjoint $\Phi_s: Gr_{n+1} \longrightarrow Gr_{k+1}$ it is shown in [6] that in Gr_n epimorphisms and surjective homomorphisms coincide. Thence the category of n-groups is a perfect category.

The fact that in an n-group the identity element is usually missing leads to the idea that Gr_n is not a connex category (cf. [3]).

3. <u>Proposition 1.</u> If $(G,o) \in Gr_n$ then

1. there exists a bijective function between the n-subgroups of G and subobjects of G
2. there exists a bijective function between the quotient-groups of G and quotient objects of G.

<u>Proof</u>. Let $S'(G)$ be the set of n-subgroups of G, $S(G)$ - the class of subobjects of G, $C(G)$ - the set of quotient-groups of G, $Q(G)$ - the class of quotient objects of G.

1. We prove that the function $f: S'(G) \to S(G)$, $f(H)=[H,i]$ is a bijective one, where $i: H \to G$, $i(x)=x$ is the inclusion homomorphism; i is an injective homomorphism i.e. monomorphism. If $f(H_1)=f(H_2) \Rightarrow [H_1,i_1]=[H_2,i_2]$ there exists an isomorphism ξ such that $i_1=i_2 o \xi$, i.e. $\xi(x)=x$, for any $x \in H_1$. As ξ is surjective it follows that ξ is the identity homomorphism; thus $H_1=H_2$. Hence f is injective.

If $[U,\mu]$ is a subobject of G then μ is a monomorphism i.e. injective homomorphism, $\xi: U \longrightarrow \mu(U)$, $\xi(x)=\mu(x)$ is a bijective homomorphism, i.e. isomorphism, and $\mu=i o \xi$. Thence $(\mu(U),i) \in [U,\mu]$. $\mu(U)$ is a homomorphic image of an n-group, hence is an n-subgroup of G and therefore $[U,\mu]=f(\mu(u))$. We proved that f is also surjective.

2. We prove that the function $g: C(G)\to Q(G)$, $g(G/_{\equiv})=[p,G/_{\equiv}]$, where "$\equiv$" is a congruence relation in G, and $p:G\to G/_{\equiv}$, $p(a)=\langle a\rangle$ is the canonical homomorphism; p is a surjective homomorphism i.e. epimorphism.

If $g(G/_{\equiv_1})=g(G/_{\equiv_2}) \Rightarrow [p_1,G/_{\equiv_1}]=[p_2,G/_{\equiv_2}] \Rightarrow$ there exists an isomorphism $\xi: G/_{\equiv_2}\to G/_{\equiv_1}$ such that $p_1=\xi o p_2$. As ξ is a bijective homomorphism it follows: $a \overset{1}{\equiv} b \Leftrightarrow p_1(a)=p_1(b)\Leftrightarrow$ $\Leftrightarrow(\xi o p_2)(a)=(\xi o p_2)(b) \Leftrightarrow \xi(p_2(a))=\xi(p_2(b))\Leftrightarrow p_2(a)=$ $=p_2(b)\Leftrightarrow a \overset{2}{\equiv} b$, thence $G/_{\equiv_1}=G/_{\equiv_2}$. Hence g is injective.

If $[\mu,U]$ is a quotient object of G, then is an epimorphism i.e. surjective homomorphism and there exists an isomorphism $\xi: G/_{\overset{\mu}{\equiv}}\to U$ such that $\mu=\xi o p$, where $p:(G,o)\to(G/_{\overset{\mu}{\equiv}}, *)$ is the canonical homomorphism ($a\overset{\mu}{\equiv}b \Leftrightarrow \mu(a)=\mu(b)$). This yields to: $(p,G/_{\overset{\mu}{\equiv}})\in[\mu,U]$ and, knowing that $G/_{\overset{\mu}{\equiv}}$ is a quotient - n-group of G, it follows that $g(G/_{\overset{\mu}{\equiv}})=[\mu,U]$, hence g is surjective.

<u>Corollary 1</u>. Gr_n is a local and colocal small category.

<u>Proposition 2</u>. Gr_n is a category with images and coimages.

<u>Proof</u>. Take $G,H\in Gr_n$ and $f\in Hom(G,H)$. We prove that $Imf=[f(G),i]$ and $Coim\ f=[p,G/\underset{f}{\equiv}]$.

We prove that $[f(G),i]\in S(H)$ and $[f(G),i]$ is the image of f (see [2]). i is an injective homomorphism i.e. monomorphism, $f:G\to H$ is a homomorphism thence $f(G)\in S'(H)$ and, by Proposition 1, it follows that $[f(G),i]\in S(H)$.

We prove that $[f(G),i]$ fulfills the conditions to be an image to f.

1. Obviously $f_1:G\to f(G)$, $f_1(x)=f(x)$ is a homomorphism, a surjective one, and $f=iof_1$.

2. If $[U,\mu]\in S(H)$ and there exists $f_2:G\to U$ such that $f=\mu o f_2$

then, knowing that f_1 is an epimorphism (right inversible), it follows that $i = \mu \circ (f_2 \circ f_1^{-1})$ and therefore $[f(G), i] \leq [U, \mu]$. Thus $\mathrm{Im} f = [f(G), i]$.

We prove that $[p, G/\mathcal{E}] \in Q(G)$ and $[p, G/\mathcal{E}]$ is the coimage of f. p is a surjective homomorphism i.e. epimorphism in Gr_n; $(G/\mathcal{E}, *) \in \mathcal{C}(G)$ and, by Proposition 1, we have $[p, G/\mathcal{E}] \in Q(G)$. We prove that $[p, G/\mathcal{E}]$ fulfills the conditions to be a coimage to f.

1. There exists an isomorphism $\bar{f}$ (because "$\mathcal{E}$" is a congruence relation on G) such that $f = (i \circ \bar{f}) \circ p$.

2. Take $\xi = i \circ \bar{f}$. If $[\mu', U'] \in Q(G)$ and there exists $f_3 : U' \to H$ such that $f = f_3 \circ \mu'$ then, knowing that $\bar{f}$ is a monomorphism (left inversible), it follows that $p = (\bar{f}^{-1} \circ f_3) \circ \mu$ and therefore $[p, G/\mathcal{E}] \leq [\mu', U']$.

Thus the proof is complete.

<u>Proposition 3</u>. Gr_n is a category with pullbacks.

<u>Proof</u>. Take $(G_1, \circ)$, $(G_2, *)$, $(G, O) \in Gr_n$. We prove that for any pair of morphisms $\alpha_1 \in \mathrm{Hom}(G_1, G)$, $\alpha_2 \in \mathrm{Hom}(G_2, G)$ there exist (P, β_1, β_2) such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\beta_1} & G_1 \\ {\scriptstyle \beta_2}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ G_2 & \xrightarrow[\alpha_2]{} & G \end{array}$$

is commutative.

Let $(G_1 \times G_2, \bar{o}) \in Gr_n$ be the direct product of G_1 and G_2, noted $G_1^o \circledast G_2^*$ defined by

$((g_1, h_1), (g_2, h_2), \ldots, (g_n, h_n))_{\bar{o}} = ((g_1, g_2, \ldots g_n)_o,$

$(h_1, h_2, \ldots, h_n)_*)$ and take $P = \{(g, h) \in G_1^o \circledast G_2^* \mid \alpha_1(g) = \alpha_2(h)\}$

which, it is easy to prove, is an n-subgroup of $G_1^o \circledast G_2^*$.

We introduce $\beta_1 : P \to G_1$, $\beta_1((g, h)) = g$; $\beta_2 : P \to G_2$, $\beta_2((g, h)) = h$. They are the restrictions to P of the canonical transformations and they are homomorphisms.

We prove that the such defined (P, β_1, β_2) is pullback of the pair α_1, α_2.

Indeed, for any $(g,h) \in P$ we have: $(\alpha_1 \circ \beta_1)((g,h)) = \alpha_1(g) = \alpha_2(h) = (\alpha_2 \circ \beta_2)((g,h))$ thence the first condition is fulfilled.

Take $P' \in Gr_n$ and two homomorphisms $\beta_1' : P' \to G_1$, $\beta_2' : P' \to G_2$ such that $\alpha_1 \circ \beta_1' = \alpha_2 \circ \beta_2'$. Then, for any $x \in P'$ we have $\alpha_1(\beta_1'(x)) = \alpha_2(\beta_2'(x))$, whence $(\beta_1'(x), \beta_2'(x)) \in P$. Thus we are able to define the function $\gamma : P' \to P$, $\gamma(x) = (\beta_1'(x), \beta_2'(x))$ which is a homomorphism of n-groups. We prove that $\beta_1' = \beta_1 \circ \gamma$ and $\beta_2' = \beta_2 \circ \gamma$. For any $x \in P'$ we have $(\beta_1 \circ \gamma)(x) = \beta_1((\beta_1'(x), \beta_2'(x))) = \beta_1'(x)$ and $(\beta_2 \circ \gamma)(x) = \beta_2((\beta_1'(x), \beta_2'(x))) = \beta_2'(x)$.

We still have to prove that γ is unique.

Let $\gamma' : P' \to P$ be a homomorphism such that $\beta_1' = \beta_1 \circ \gamma'$ and $\beta_2' = \beta_2 \circ \gamma'$. That means: for any $x \in P'$, $\beta_1'(x) = \beta_1(\gamma'(x))$ and $\beta_2'(x) = \beta_2(\gamma'(x))$. Then, if we note $\gamma'(x) = (g,h)$ we have $\beta_1'(x) = g$; $\beta_2'(x) = h$ and thence $\gamma(x) = (\beta_1'(x), \beta_2'(x)) = (g,h) = \gamma'(x)$. Thus $\gamma = \gamma'$, i.e. γ is unique.

The proof is now complete; (P, β_1, β_2) is the pullback of the pair α_1, α_2.

<u>Corollary 2</u>. If $\alpha_1 : G_1 \to G$, $\alpha_2 : G_2 \to G$ are injective homomorphisms of n-groups then there exists the intersection of subobjects of G, $[G_1, \alpha_1]$, $[G_2, \alpha_2]$ and $[G_1, \alpha_1] \cap [G_2, \alpha_2] = [P, \alpha]$; where $\alpha = \alpha_1 \circ \beta_1 = \alpha_2 \circ \beta_2$.

The corollary proves that intersection of two n-subgroups G_1, G_2 of an n-group G is also an n-subgroup of G which is isomorphic to P.

<u>Proposition 4</u>. Gr_n is a category with generators and free objects. Gr_n is also a concrete category and the functor

$H^2 = \mathrm{Hom}_{Gr_n}(\mathbb{Z}, -)$ is a faithful functor. (see [3]).

Proposition 5. Gr_n is a category with pushouts.

The pushout of a pair of homomorphisms $\alpha_1 \in \mathrm{Hom}(G, G_1)$, $\alpha_2 \in \mathrm{Hom}(G, G_2)$ is isomorphic to an n-subgroup of the free product of the n-groups G_1 and G_2, defined as in [5].

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