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DISTINGUISHED SYSTEM OF COSET REPRESENTATIVES AND NORMAL COMPLEMENT OF A SUBGROUP OF FINITE GROUP

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1. Introduction

Suppose that H is a subgroup of the finite group G and that N is a normal subgroup of H / in symbols $N \trianglelefteq H \leq G$ /.

A normal subgroup K of G is a normal complement of H in G over N if $G = HK$, $H \cap K = N$.

A system R of /right/ coset representatives of H in G is distinguished over N if $h^{-1}NRh = NR$ for every $h \in H$./ It is evident that the condition $h^{-1}NRh = NR$ is equivalent with $h^{-1}Rh \subseteq NR$ for every $h \in H$./

In particular case a normal complement of H in G over $\{1\}$ is called simply normal complement of H in G and a system of coset representatives of H in G which is distinguished over $\{1\}$ is a distinguished system of coset representatives of H in G.

Assume that K is a normal complement of H in G over N and that Nx_i / $1 \leq i \leq n$ / is the set of all right cosets of N in K. Then Hx_i / $1 \leq i \leq n$ / is the set of all right cosets of H in G and the system $R = \{x_1, x_2, \ldots, x_n\}$ of right coset representatives of H in G has the property that $NR = K$. Hence $h^{-1}NRh = NR$ for every $h \in H$. Thus R is distinguished over N. This proves the following fact:

The existence of normal complement of H in G over N implies the existence of system of coset representatives of H in G which is tistinguished over N.

In the particular case $N = \{1\}$ there are many results which assert that the existece of distinguished system R of cose representatives of H in G, under various conditions on H and R implies the existence of normal complement of H in G. I mention here the results of papers of R.Kochendörffer [1], M. Suzuki [2] and G. Zappa [3].

The aim of this paper is to study, in the general case $N \trianglelefteq H \leq G$, the existence of normal complement of H in G over N under the condition of existence of system of coset representatives of H in G which is distinguished over N and under various conditions on H, N and on system.

2. Preliminary results

LEMMA 1. Let R be a system of right coset representatives of the subgroup H of G and

$$D(H,R) = \{N \trianglelefteq H \mid R \text{ is distinguished over } N\}.$$

D(H,R) is a filter in the lattice of normal subgroup of H / i.e. if $N \in D(H,R)$ and $N \leq N_1 \trianglelefteq H$, then $N_1 \in D(H,R)$; if $N_1, N_2 \in D(H,R)$ then $N_1 \cap N_2 \in D(H,R)$ ./

Proof. The statements follows directly from the definition of the system of coset representatives of H in G, which is distinguished over a normal subgroup of H.

LEMMA 2. Let HxH be a double coset of the subgroup H of G and let S be a system of right coset representatives of $H \cap x^{-1}Hx$ in H. Then

$$HxH = \bigcup_{s \in S} Hs^{-1}xs \quad \text{/disjoint/}$$

Proof. If $Hs_1^{-1}xs_1 = Hs_2^{-1}xs_2$, $s_1, s_2 \in S$, then $xs_1s_2^{-1}x^{-1} \in H$. Hence $s_1s_2^{-1} \in H \cap x^{-1}Hx$. Therefore $s_1 = s_2$. It follows that the number of right cosets of H in $\bigcup_{s \in S} Hs^{-1}xs$ is $|H : H \cap x^{-1}Hx|$. This implies that

$$HxH = \bigcup_{s \in S} Hs^{-1}xs \quad \text{/disjoint/.}$$

<u>THEOREM 1.</u> If $N \trianglelefteq H \leq G$, then the following conditions are equivalent:

(I) There exists a system R of coset representatives of H in G which is distinguished over N.

(II) There exists a system T of double coset representatives of H in G, such that $H \cap x^{-1}Hx \subseteq N_G(Nx)$ for every $x \in T$. /$N_G(Nx)$ is the normalizer of Nx in G/

(III) There exists a system T of double coset representatives of H on G, such that $H \cap x^{-1}Hx = N_H(Nx)$ for every $x \in T$.

<u>Proof.</u> Assume that R is a system of coset representatives of H in G which is distinguished over N. Suppose that $T \subseteq R$ is a system of double coset representatives of H in G. If $x \in T$ and $h \in H \cap x^{-1}Hx$, then $h = x^{-1}kx, k \in H$. Hence $h^{-1}xh = h^{-1}kx \in Hx$. It follows by (I) that $h^{-1}xh \in h^{-1}Rh \cap NR$. Therefore $h^{-1}xh \in NR \cap Hx$. Thus $h^{-1}xh \in Nx$. This implies that $Nx = Nh^{-1}xh = h^{-1}Nxh$, i.e. $h \in N_G(Nx)$. Hence (I) implies (II).

Suppose that $T = \{x_1, x_2, \ldots, x_m\}$ is a system of double cose representatives of H in G, such that $H \cap x_i^{-1}Hx_i \subseteq N_G(Nx_i)$ / $1 \leq i \leq m$ /.

If $h \in N_H(Nx_i)$, then $h^{-1}Nx_ih = Nx_i$, i.e. $Nh^{-1}x_ih = Nx_i$. It follows that $Hh^{-1}x_ih = Hx_ih = Hx_i$. Hence $x_ihx_i^{-1} \in H$. This implies that $h \in H \cap x_i^{-1}Hx_i$. Therefore $N_H(Nx_i) \subseteq H \cap x_i^{-1}Hx_i$. The inverse inclusion follows from (II). Thus (II) implies (III).

Assume that $T = \{x_1, x_2, \ldots, x_m\}$ is a system of double coset representatives of H in G, such that $H \cap x_i^{-1}Hx_i = N_H(Nx_i)$ / $1 \leq i \leq m$ /. Let S_i be a system of right coset representatives of $H \cap x_i^{-1}Hx_i = N_H(Nx_i)$ in H and we consider the sets $R_i = \{s_i^{-1}x_is_i \mid s_i \in S_i\}$ / $1 \leq i \leq m$ /. Then $h^{-1}NR_ih = NR_i$ for every $h \in H$ and it follows by Lemma 2, that $Hx_iH = HR_i = \bigcup_{s_i \in S_i} Hs_i^{-1}x_is_i$ /disjoint/. Therefore $\bigcup_{i=1}^{m} R_i$ is a system of right coset representatives of H in G and $h^{-1}N\left(\bigcup_{i=1}^{m} R_i\right)h = N\left(\bigcup_{i=1}^{m} R_i\right)$, i.e. $\bigcup_{i=1}^{m} R_i$ is distinguished over N. Thus (III) implies (I).

COROLLARY 1./ Lemma 1. and Lemma 2. of [3]/ If H is a subgroup of group G, then the following conditions are equivalent:

(i) There exists a distiguished system of coset representatives of H in G.

(ii) There exists a system T of double cose representatives of H in G, such that $H \cap x^{-1}Hx \subseteq C_G(x)$ for every $x \in T$ / $C_G(x)$ is the centralizer of x in G/.

(iii) There exists a system T of double coset representatives of H in G, such that $H \cap x^{-1}Hx = C_H(x)$ for every $x \in T$.

3. The case H/N abelian

THEOREM 2. Assume that $N \trianglelefteq H \leq G$, H/N is abelian and that there exists a system R of coset representatives of H in G which is distinguished over N.

(I) If there exists a prim p, such that $|H:N|$ is divisible by p, but $|G:H|$ is not divisible by p, then G has a normal subgroup K, such that $|G:K|$ is divisible by p.

(II) If $|G:H|$ and $|H:N|$ are relatively prime, then there exists a normal complement of H in G over N.

Proof. Let $t: G \to H/N$ be the transfer of G in H. If $h \in H$, then there exists a subset $\{x_1, x_2, \dots, x_r\}$ of R and $n_i \in \mathbb{N}$ / $1 \leq i \leq r$ /, such that

$$t(h) = N\left(\prod_{i=1}^{r} x_i h^{n_i} x_i^{-1}\right), \quad \sum_{i=1}^{r} n_i = |G:H|$$

and n_i is minimal such that

$$x_i h^{n_i} x_i^{-1} \in H \quad / 1 \leq i \leq r /. [4]$$

It follows that $Hx_i = Hh^{-n_i}x_i h^{n_i}$. Since R is distinguished over N, it results that $h^{-n_i}x_i h^{n_i} \in NR$. Therefore $Nh^{-n_i}x_i h^{n_i} = Nx_i$, i.e. $Nx_i h^{n_i} x_i^{-1} = N h^{n_i}$. Thus

$$t(h) = N h^{|G:H|}$$

If $p \mid |H:N|$ and $p \nmid |G:H|$, then there exists a p-element $h_0 \in H-N$. It follows that $t(h_0) = Nh_0^{|G:H|} \neq N$. Hence the kernel K of t is a proper normal subgroup of G and $p \mid |G:K|$, Therefore (I) holds.

If $|G:H|$ and H:N are relatively prime and $h \in H$, then $t(h) = Nh^{|G:H|} = N$ if and only if $h \in N$. It follows that if K is the kernel of t, then $H \cap K = N$ and $t(G) = H/N$. Hence $G = HK$, $H \cap K = N$, i.e. K is a normal complement of H in G over N. Therefore (II) is true.

<u>COROLLARY 2.</u> If $N \trianglelefteq H \leq G$, H/N is abelian, suppose that there is a system T of double coset representatives such that $H \cap x^{-1}Hx = N_H(Nx)$ for every $x \in T$.

(I) If p is a prime such that $p \mid |H:N|$ and $p \nmid |G:H|$, then G has a normal subgroup K such that $p \mid |G:K|$.

(II) If $|G:H|$ and $|H:N|$ are relatively prime, then G has normal complement of H over N.

<u>Proof</u>. The statements follows directly from the Theorem 1 and Theorem 2.

<u>COROLLARY 3</u>. Suppose that $N \trianglelefteq H \leq G$ $(N \neq H)$ and that there exists a system of coset representatives of H in G which is distinguished over N/ or there exists a system T of double coset representatives of H in G such that $H \cap x^{-1}Hx = N_G(Nx)$ for every x T /.

If H/N is solvable, $|G:H|$ and $|H:N|$ are relatively prime, then G has a proper normal subgroup K such that $G = HK$, $N \subseteq H \cap K$.

<u>Proof</u>. Since H/N is solvable $(N \neq H)$, it follows that there is a proper normal subgroup N_1 of H such that $N \subseteq N_1$ and H/N_1 is abelian. It is clear that $|G:H|$ and $|H:N_1|$ are relatively prime. Hence, by Theorem 2 / by Corollary 2/ and Lemma 1, there exists a normal complement K of H in G over N_1, i.e. $G = HK$, $N \subseteq N_1 = H \cap K$.

<u>COROLLARY 4</u>. If H is a Sylow subgroup of G and there exists

a system of coset representatives of H in G which is distinguished over a proper normal subgroup N of H/ or there exists a system T of double coset representatives of H in G such that $H \cap x^{-1}Hx = N_H(Nx)$ for every $x \in T$/, then G has a proper normal subgroup K such that $G = HK$, $N \subseteq H \cap K$.

Proof. Since H is a Sylow subgroup of G and N is a proper normal subgroup of H, there exists a proper normal subgroup N_1 of H such that $N \subseteq N_1$ and H/N_1 is abelian. It is clear that $|G:H|$ and $|H:N|$ are relatively prime. Hence by Corollary 3 there exists a normal complement K of H in G over N_1, i.e. $G = HK$ and $N \subseteq H \cap K$.

R E F E R E N C E S

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