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THE STARLIKENESS OF A PARTICULAR CLASS OF INTEGRAL OPERATORS

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Let A be the class of all analytic functions f in the unit disc $U=\{z\in\mathbb{C} / |z|<1\}$, with $f(0)=0$ and $f'(0)=1$.

A function $f\in A$ is said to be starlike if $\mathrm{Re}[zf'(z)/f(z)]>0$ in U. We denote the class of starlike functions by S^*.

Let $g\in A$ with $g(z)/z\neq 0$ in U and let c be complex number with Re $c>0$. Consider the integral operators J_{g^c} on A defined by $F = J_{g^c}(f)$, where

$$(1)\qquad F(z) = \frac{c+1}{[g(z)]^c}\int_0^z f(t)\,\frac{[g(t)]^c}{t}\,dt \quad , \quad z\in U\ ,\ f\in A\ .$$

Making the substitution $w=uz$, (1) can be rewritten:

$$F(z) = (c+1)\left[\frac{z}{g(z)}\right]^c \int_0^1 f(uz)\left[\frac{g(uz)}{uz}\right]^c u^{c-1}du \quad ,$$

where all powers are the principale ones. This shows that the integral operators J_{g^c} is well defined.

In this article we employ the method of admissible functions due to P.T.Mocanu and S.S.Miller [1] , [2] , [4] , to obtain sufficient conditions on g , so that $J_{g^c}(S^*)\subset S^*$.

We will need the following two lemmas to prove our main results.

Lemma1. [3] Let c be complex number and $f\in A$. If $\mathrm{Re}[zf'(z)/f(z)+c]>0$ in U , then the function F defined by

$$F(z) = \frac{1+c}{z^c}\int_0^z f(t)t^{c-1}dt$$

belongs to A and $F(z)/z\neq 0$ in U .

Lemma2. [4] Let the function $\psi:\mathbb{C}^2\times U\to\mathbb{C}$ satisfy the condition

(2) $\qquad \mathrm{Re}\,\psi(ix,y;z)>0$

for all real x and y with $y\le -(1+x^2)/2$ and all $z\in U$. If p is analytic in U , $p(0)=1$ and $\mathrm{Re}\,\psi(p(z),zp'(z);z)>0$ for $z\in U$, then $\mathrm{Re}\,p(z)>0$ in U.

Theorem. Let $g\in A$ and c be a complex number with Re $c>0$. If $B(z) = zg'(z)/g(z)$ satisfies:

(3) $\qquad \mathrm{Re}[cB(z)]>0$

and

(4) $\operatorname{Re}[cB(z)]\operatorname{Re}[cB(z)-2|c|^2\overline{B(z)}zB'(z)] \geq [\operatorname{Im}(czB'(z))]^2$

in U , then $J_{g^c}(S^*)\subset S^*$.

Proof. Let $f\in S^*$ and $h(z)z^{c-1} = f(z)[g(z)]^c/z$. We have

$$\operatorname{Re}\left[\frac{zh'(z)}{h(z)} + c\right] = \operatorname{Re}\left[\frac{zf'(z)}{f(z)} + c\frac{zg'(z)}{g(z)}\right] > 0$$

and by Lemma 1. the function

$$H(z) = \frac{c+1}{z^c}\int_0^z h(t)t^{c-1}dt$$

satisfies $H(z)/z\neq 0$ in U . Since $g(z)/z\neq 0$ and $F(z) = g(z)/z^{-c}H(z)$, we deduce $F(z)\neq 0$. Hence the function $p(z)=zF'(z)/F(z)$ is analytic in U and $p(0)=1$.

From (1) we easily obtain:

(5) $$p(z) + \frac{zp'(z)+czB'(z)}{p(z)+cB(z)} = \frac{zf'(z)}{f(z)} .$$

Let

$$\psi(u,v;z) = u + \frac{v+czB'(z)}{u+cB(z)} .$$

Since $f\in S^*$, from (5) we deduce $\operatorname{Re}\psi(p(z),zp'(z);z)>0$ for $z\in U$. In order to use Lemma 2 , we must verify that the function ψ satisfies the condition (2). We have:

$$\operatorname{Re}\psi(ix,y;z) = \operatorname{Re}\frac{y+czB'(z)}{ix+cB(z)} =$$

$$= \frac{y\operatorname{Re}[cB(z)]+|c|^2\operatorname{Re}[zB'(z)\overline{B(z)}] + x\operatorname{Im}[czB'(z)]}{|cB(z) + ix|^2} .$$

Since $\operatorname{Re}[cB(z)]>0$, for $y\leq -(1 + x^2)/2$ we obtain:

$$\operatorname{Re}\psi(ix,y;z)\leq \frac{\alpha+2\beta x+\gamma x^2}{2|cB(z)+ix|^2}$$

where $\alpha = 2|c|^2\operatorname{Re}[zB'(z)\overline{B(z)}]- \operatorname{Re}[cB(z)]$, $\beta= \operatorname{Im}[czB'(z)]$ and $\gamma=-\operatorname{Re}[cB(z)]$.

Since the condition (4) is equivalent to $\beta^2-\alpha\gamma\leq 0$, we deduce that $\operatorname{Re} p(z) = \operatorname{Re}[zF'(z)/F(z)] > 0$, which shows that $F\in S^*$.

In the particular case c=1 this theorem was proved by S.S.Miller and P.T.Mocanu [5] .

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