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A NOTION OF ALMOST EVERYWHERE PRIMITIVE

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Various generalizations of the usual notions of derivative and integral are known, see [5], [8]. For example, interesting generalizations of the descriptive definition for nonabsolutely convergent integrals were given in [9], [11], [12], [13], [14].

The classes of primitives for this integrals are, generally, of continuous functions [9], [11]. However, the classes of primitives for the integrals of Ellis [11] and the integrals of Lee [23] are not classes of continuous functions.

In order to be able to integrate "very discontinuous" functions, in this paper we introduce a notion of discontinuous but simmetricaly approximate continuous primitive, by replacing the usual derivative by the approximate derivative.

The starting point of this notion is an elementary problem of analysis [1].
The obtained entity, a discontinuous primitive, may be used to define a nonabsolute integral [2], which includes the Riemann, Lebesgue and Denjoy-Perron (generalized Riemann) integrals as particular cases, because the approximate derivatives possess all the known important properties of derivatives, see [7], and the simmetrically continuity assure the uniqueness of our integration.

Recall that a statement in terms of points in a subset of $\mathbb{R}$ is said to hold almost everywhere (abbreviated a.e.) if it is true except at the points of a Lebesgue null set (of Lebesgue measure zero).

Suppose that E_x is a set having density 1 at $x \in \mathbb{R}$. Then the function $F:\mathbb{R} \to \mathbb{R}$, is approximately differentiable at x if the limit

$$\lim_{\substack{y \to x \\ y \in E_x}} \frac{F(y)-F(x)}{y-x}$$

exists.

This limit is then the approximate derivative of F at the point x.

We denote, as usually, the approximate derivative by F'_{ap}.

If the limit exists for almost everywhere x, then F is approximately differentiable a.e.

Throughout this paper we denote by I an interval in the real line and by D a Lebesgue null set, $D \subset I$.

It is clear that, for every point $x \in I$, $I \setminus D$ is a set having density 1 at x.

Also, in the sequel, the definitions will be given by replacing the "limit" by "approximate limit", without special mention.

DEFINITION 1.

Let $f:I \to \mathbb{R}$ be given.

A function $F:I\setminus D \to \mathbb{R}$ is an <u>a.e. primitive of f on I</u> provided

(i) F is approximately differentiable on $I \setminus D$;

(ii) $F'_{ap}(x)=f(x)$, for all $x \in I\setminus D$ ($F'_{ap}=f$ a.e. on I);

(iii) For each $d \in \mathring{D}_{\Pi}$, there exists the following limit

$$\lim_{\substack{\varepsilon \to 0 \\ d-\varepsilon,\, d+\varepsilon \in I\setminus D}} \left[F(d+\varepsilon) - F(d-\varepsilon)\right] = 0.$$

(if $d \in \operatorname{fr} I$, then the existence of finite $F(d+)$ or $F(d-)$ is assumed).

We say also that f is a.e. primitivable on I.

REMARKS.

1) Obviously, f need not be defined on D;

2) If F is a (strict) primitive of $f: I \to \mathbb{R}$, then F is an a.e. primitive of f on I;

(Recall that a function $F: I \to \mathbb{R}$ is a strict primitive of f on I if F is differentiable on I and $F'(x)=f(x)$, for all x in I[4]; and F is a primitive of f on I provided

(i) F is continuous on I,

(ii) $F'(x)=f(x)$, for all x in I except possibly a finite or countable infinite set of values of x [4], [25];

3) If F is an a.e. primitive of f, then $F+c$ is also an a.e. primitive of f, for any constant function c;

4) The a.e. primitive is an effective generalization of the (strict) primitive, as shown by

EXAMPLE 1.

Let $f: [0,1] \to \mathbb{R}$ be the characteristic function of the Cantor ternary set C_0, i.e. $f(x)=1$, when $x \in C_0$ and $f(x)=0$, when $x \in [0,1]\setminus C_0$. Then f does not possesses a primitive on $[0,1]$, see [4] or [27] p.365.

Since $F: [0,1]\setminus C_0 \to \mathbb{R}$,

$$F(x) = k \text{ (const)}$$

is an a.e. primitive of f on $[0,1]$, f is a.e. primitivable.

Note that the function f in Example 1 is Riemann integrable. It is well known that, generally, a Riemann integrable function is not strict primitivable [4], [27]. We have a more general result

THEOREM 1.

Let $f: I \to \mathbb{R}$ be a Henstock-Kurzweil integrable function. Then f is a.e. primitivable on I.

<u>Proof.</u>

Using the equivalence between the Henstock-Kurzweil integrability and restricted Denjoy-Perron integrability [17], it results that there exists an A C G (absolutely continuous generalized) function

$$F : I \to \mathbb{R},$$

such that

$$F' = f \text{ a.e. on } I.$$

From the absolute continuity of F we deduce that the function

$$G : I \setminus D \to \mathbb{R},$$

where D is the set of nondifferentiability of F, defined by

$$G(x) = F(x), \text{ for all } x \in I \setminus D,$$

is an a.e. primitive of f on I.

REMARK.

From the above theorem and from the following example it results that over any interval $[a,b]$ the class of Henstock-Kurzweil integrable functions is a proper subset of the class of a.e. primitivable functions.

EXAMPLE 2.

Let $f: [-1,1] \to \mathbb{R}$ be the function defined by $f(x)= \frac{1}{x}$, when $x \neq 0$ and $f(0)=0$.

Then f is not Henstock-Kurzweil integrable on both intervals $[0,1]$ and $[-1,0]$, see [25], exercise 20, p. 69, hence f is not Henstock-Kurzweil integrable on $[-1,1]$. However, $F: [-1,1] \setminus \{0\} \to \mathbb{R}$,

$$F(x) = \ln|x|,$$

is an a.e. primitive of f on $[-1,1]$, since

$$\lim_{\varepsilon \to 0} \left[F(\varepsilon) - F(-\varepsilon) \right] = 0,$$

and F being differentiable on $[-1,1] \setminus \{0\}$, F is approximately differentiable on $[-1,1] \setminus \{0\}$. Let us remark that the function in this example is "very discontinuous".

In order to obtain a monotonicity theorem for our primitive we need a corollary of O'Malley's monotonicity theorems [26], stated in [6] in terms of extreme approximate derivates: If F is measurable, $\underline{F}'_{ap} \geq 0$ a.e. and $\underline{F}'_{ap} > -\infty$ everywhere, then F is nondecreasing.

Using this result we obtain

THEOREM 2.

If F is an a.e. primitive of $f: I \to \mathbb{R}$, and $f(x) \geq 0$ for all $x \in I$, then F is nondecreasing.

REMARKS.

1) If $F: I \setminus D \to \mathbb{R}$, then it suffices that $f(x) \geq 0$, for all $x \in I \setminus D$;

2) We mention that the known differences between approximate derivates and ordinary derivates seem to involve the primitives.

For the (strict) primitive of a function f, the continuity condition imply that, if F, G are two such primitives, then

$$F - G = \text{const on } I.$$

Generally, if $F' = G'$ a.e. on I, then $F-G$ is not constant on I [17].

But the simmetrically continuity condition (iii) from definition 1 implies the following

COROLLARY.

Let $F, G: I \setminus D \to \mathbb{R}$ be two a.e. primitives of a given function f.

Then

$$F - G = \text{const on } I \setminus D.$$

<u>Proof.</u> Obviously,

on every interval $I_1 \subset I \setminus D$ we have

$$F - G = \text{const},$$

since, in this case, the restrictions of both F and G to I_1 are strict primitives of $f|_{I_1}$.

Hence, it sufficies to show that, if

$$F-G = c_1 \text{ on } E_1 \subset I \setminus D,$$

$$F-G = c_2 \text{ on } E_2 \subset I \setminus D,$$

where

$$E_1 = (-\infty, d) \cap I \setminus D,$$

$$E_2 = (d, +\infty) \cap I \setminus D,$$

and

$$d \in D,$$

then, necessarly, $c_1 = c_2$.

Indeed,

$$c_2 - c_1 = \lim_{\substack{\varepsilon \to 0 \\ d+\varepsilon,\, d-\varepsilon \in I \setminus D}} \left[(F-G)(d+\varepsilon) - (F-G)(d-\varepsilon) \right]$$

$$= \lim \left[F(d+\varepsilon) - F(d-\varepsilon) \right] - \lim \left[G(d+\varepsilon) - G(d-\varepsilon) \right] = 0.$$

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