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2-CONTINUOUS FUNCTIONS

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FUNCTII 2-CONTINUE
(rezumat)

In această notă se arată că mulţimea funcţiilor 2-continue este intermediară între mulţimea funcţiilor continue uniform şi mulţimea funcţiilor continue punctual.

In this note it is shown that the set of 2-continuous functions is situated between the set of uniformly continuous functions and the set of continuous functions.

First we establish a lemma concerning the continuity of a function.

LEMMA. The function $f:(X,\mathcal{T}) \longrightarrow (Y,\mathcal{V})$ is continuous if and only if $pr_1:((G,\mathcal{T}\times\mathcal{V}|_G) \longrightarrow (X,\mathcal{T})$ is open, where $G = \{(x,f(x)) / x \in X\}$ and pr_1 is the first projection.

<u>Proof.</u> Let f be continuous and $U \times V \in \mathcal{T}\times\mathcal{V}$ an element of the base of the product topology. We have $pr_1((U\times V)\cap G) = \{x \in U / f(x) \in V\} = U \cap f^{-1}(V) \in \mathcal{T}$, since f is continuous; therefore the image under pr_1 of every open set from $\mathcal{T}\times\mathcal{V}|_G$ is open. It follows that pr_1 is open. Conversely, let V be an open subset of Y. Then $f^{-1}(V) = X \cap f^{-1}(V) = pr_1((X \times V)\cap G) \in \mathcal{T}$ since pr_1 is open. We conclude that f is continuous.

Let X,Y be nonempty sets, $f:X \longrightarrow Y$ a function and β, β' topologies on X^2, respectively Y^2. The function $f \times f : X^2 \longrightarrow Y^2$, $(f \times f)(x,y) = (f(x),f(y))$ is called the binary function of f. We say that $f:(X,\beta) \longrightarrow (Y,\beta')$ is 2-continuous if $f \times f$ is continuous [2] . The definition of A.A.Ivanov [2] concerning 2-continuous functions introduces a type of continuity of a function on spaces without topologies, but one can provide arguments from General Topology in order to motivate this definition. Here are three such arguments.

1) The function $f:(X, \mathcal{T}) \longrightarrow (Y, \mathcal{U})$ is continuous if and only if $f\times f:(X^2, \mathcal{T}^2) \longrightarrow (Y^2, \mathcal{U}^2)$ is continuous. But on X^2 there exist many topologies which are not the product of two topologies on X (indecomposable topologies). For example if $X = \{a,b,c\}$, on X^2 there exist 841 decomposable topologies and more than 63.10^9 indecomposable ones [1] . Thus the study of 2-continuous functions is imposed by the great number of these functions.

2) The concept of uniform continuity of a function $f:(X,\mathcal{U}) \longrightarrow (Y,\mathcal{U}')$ is defined using the binary function.

3) By virtue of the above lemma, the continuity of a function can be defined using a topology on a subset of X^2.

In order to establish a connection between 2-continuous functions and continuous functions we must define a topology on X if there exists a topology β on X^2. We shall indicate two procedures for obtaining a topology on X by means of a topology β on X^2. The first procedure has been indicated by A.A.Ivanov; the second one is considered here for the first time.

Let $a \in X$ and let $\mathcal{T}(a,\beta)$ be the topology on X determined by $\beta|_{\{a\}\times X}$, that is

$$\mathcal{T}(a,\beta) = \{G\subset X/ \ \{a\} \times G\in\beta|_{\{a\}\times X}\},$$

Hence, to a topology β on X^2 corresponds a family of topologies on X, determined by the elements of X.

<u>Theorem 1.</u> If $f:(X,\beta) \longrightarrow (Y,\beta')$ is 2-continuous and $b = f(a)$, then $f:(X, \mathcal{T}(a,\beta)) \longrightarrow (Y, \mathcal{T}(b,\beta'))$ is continuous.

<u>Proof.</u> If $f\mathrm{x}f:(X^2,\beta) \longrightarrow (Y^2,\beta')$ is continuous, then its restriction to the subset $\{a\} \times X$ is also continuous. Let $V \in \mathcal{T}(b,\beta')$, that is $\{b\} \times V \in \beta'|_{\{b\} \times Y}$. We have $(f\mathrm{x}f)^{-1}(\{b\} \times V) = \{a\} \times f^{-1}(V) \in \beta|_{\{a\} \times X}$. Hence $f^{-1}(V) \in \mathcal{T}(a,\beta)$. It follows that f is continuous.

The second procedure for obtaining a topology on X from a topology β on X^2 is the following. Let $f:X \longrightarrow X$ be an arbitrary function, G its graph and $\beta|_G$ the induced topology on G. We denote by $\mathcal{T}(f,\beta)$ the final topology determined by the first projection $pr_1:(G,\beta|_G) \longrightarrow X$. Hence, to a topology on X^2 corresponds a family of topologies on X, determined by the set of the functions from X into X. Let $I: X \longrightarrow X$ be the identity function and $\mathcal{T}(\beta) = \mathcal{T}(I,\beta)$.

<u>Theorem 2.</u> If $f:(X,\beta) \longrightarrow (Y,\beta')$ is 2-continuous, then $f:(X, \mathcal{T}(\beta)) \longrightarrow (Y, \mathcal{T}(\beta'))$ is continuous.

<u>Proof.</u> Let $V \in \mathcal{T}(\beta')$, that is $V = pr_1(B' \cap \Delta_Y)$ where $B' \in \beta'$ and $\Delta_Y = \{(y,y)/y \in Y\}$. We have $f^{-1}(V) = pr_1(f\mathrm{x}f)^{-1}(B' \cap \Delta_Y) = pr_1 B \cap \Delta_X \in \mathcal{T}(\beta)$ since $B = (f\mathrm{x}f)^{-1}(B') \in \beta$, and $\Delta_X = \{(x,x)/x \in X\}$. It follows that f is continuous.

Let U be a uniformity on X and $\beta(U) = \mathcal{T}^2(U)$ where $\mathcal{T}(U)$ is the uniform topology on X.

<u>Theorem 3.</u> If $f:(X,U) \longrightarrow (Y,U')$ is uniformly continuous, then $f:(X, \beta(U)) \longrightarrow (Y, \beta(U'))$ is 2-continuous.

R E F E R E N C E S

1 Comtet, L., Advanced Combinatorics, Dordecht-Boston, 1974.

2 Ivanov, A.A., Structuri topologhiveseovo tipa, Issledovania po topologhii III, (1973), p.5-62.

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