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NUMERICAL SOLUTION OF DIFFERENTIAL EQUATION OF n^{th} ORDER WITH DEVIATING ARGUMENT BY SPLINE FUNCTIONS

Haydar AKCA and Gh. MICULA

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Abstract. A procedure for obtaining spline function Approximations for the solutions of the equation $y^{(n)} = f(t, y(t), y(g(t)))$ is presented.

1. Introduction. Micula gave in [3] a method for obtaining the approximate solutions of the following differential equations by spline functions.

$$y^{(n)} = f(x, y)$$

with the initial conditions:

$$y(0) = y_0 \ , \ y'(0) = y_0' \ , \dots , \ y^{(n-1)}(0) = y_0^{(n-1)}$$

where $f : [\, 0, a \,] \times R \rightarrow R$, $a > 0$ is a continuous function.

The purpose of the present study is to extend the results of [3] and [4] from the ordinary case to the deviating argument differential equations.

2. Description of the spline approximation method. Consider the following initial value problem for the deviating argument differential equation

$$y^{(n)} = f(t,\ y(t),\ y(g(t))),\quad t\in[0,T] \qquad (1)$$

$$y(t) = \phi(t),\ y'(t) = \phi'(t),\ \ldots,\ y^{(n-1)}(t) = \phi^{(n-1)}(t),\quad t\in[\alpha,0],\ \alpha<0$$

where $f: [0,T]\times R^2 \to R$, $(t,u,v) \to f(t,u,v)$ is continuous in t,u,v, and also that $\phi \in C^{m-1}[\alpha,0]$, $g\in C[\alpha,T]$,

$$g'(t) \leq t-a,\quad a>0,\ t\in[\alpha,\ T].$$

Further assume that f satisfies the following Lipschitz condition

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \leq L(\|u_1-u_2\| + \|v_1 - v_2\|), \qquad (2)$$

$$(t,u_1,v_1),\ (t,u_2,v_2) \in [0,T]\times R^2$$

and $\exists\ \beta > 1$ so that

$$\|v_1 - v_2\| < \beta\,\|f(t,u_1,v_1) - f(t,u_2,v_2)\|.$$

These conditions guaranteed the existence and uniqueness of the solution $y: [\alpha,T] \to R$ of problem (1). For the qualitative behaviour of the solution y, in particular the presence of jump-discontinuities in the higher derivatives caused by the deviating function g, known as primary discontinuity. The function g does not depend on y. We shall consider the jump-discontinuities to be known for sufficiently high-order devivaties and they are disposed in the form

$$r_0 < r_1 < \ldots < r_k < \ldots < r_M$$

A polynomial spline function $s: [0,T] \to R$ of degree m and continuity class C^{m-1} $(m \geq n)$ which will be defined on each interval $[r_{k-1}, r_k]$.

Now consider the first interval $[r_0, r_1]$ which is $[0, r_1]$, divided by a uniform partition defined by the knots

$$0 = t_0 < t_1 < \ldots < t_k < \ldots < t_N = r_1,\quad t_j = jh,\ h = r_1/N.$$

On the interval $[t_0, t_1]$, the spline function s is defined by

$$s_0(t) = s_0(t_0) + \frac{s_0'(t_0)}{1!}(t-t_0) + \ldots + \frac{s_0^{(m-1)}(t_0)}{(m-1)!}(t-t_0)^{m-1} + \frac{a_0}{m!}t^m, \qquad (3)$$

with $s_0(t_0) = \phi(t_0), s_0'(t_0) = \phi'(t_0), \ldots, s_0^{(m-1)}(t_0) = \phi^{(m-1)}(t_0)$, $0 < t < h$

where the last coefficient a_0 is to be determined by requiring that s_0 should satisfy differential equation in problem (1) for $t=t_1=h$. This gives the equation

$$s_0^{(m)}(t_1) = f(t_1, s_0(t_1), s_0(g(t_1))) \quad (4)$$

Having determined the polynomial (3), on the next interval

$[t_1, t_2]$, s defined by

$$s_0(t) = \sum_{j=0}^{m-1} \frac{s_0^{(j)}(t_1)}{j!}(t-t_1)^j + \frac{a_1}{m!}(t-t_1)^m, \quad t \in [t_1, t_2] \quad (5)$$

where $s_0^{(j)}(t_1)$, $0 \leq j \leq m-1$ are left hand limits of derivatives as $t \to t_1$ of the segment of s_0 defined in (3) on $[t_0, t_1]$ and a_1 is determined as to satisfy the equation

$$s_0^{(m)}(t_2) = f(t_2, s_0(t_2), s_0(g(t_2))) \quad (6)$$

Continuing in this manner we obtain a spline function $s_0 : [r_0, r_1] \to R$ of degree m and class C^{m-1} with knots t_k, which approximates the solution y of (1) and satisfies the collocation conditions

$$s_0^{(m)}(t_{k+1}) = f(t_{k+1}, s_0(t_{k+1}), s_0(g(t_{k+1}))), \quad (k=\overline{0,N-1}) \quad (7)$$

Now consider the interval $[r_j, r_{j+1}]$, $(j=\overline{0,M-1})$ divided by a uniform partition with the points

$$\{t_k\}_{k=0}^{N}, \quad t_k = t_0 + kh, \quad k=\overline{0,N}, \quad h = (t_k - t_{k-1})/N$$

Let $\mathcal{S}_m$ denote the class of spline functions of degree $m \geq n$ and continuity class C^{m-1}, on $[0,T]$. If we denote spline function by $s \in \mathcal{S}_m$ approximating the solution of problem (1), then on the interval $[t_k, t_{k+1}]$, s is defined by

$$s(t) = \sum_{j=0}^{m-1} \frac{s^{(j)}(t_k)}{j!}(t-t_k)^j + \frac{a_k}{m!}(t-t_k)^m \quad (8)$$

$$= A_k(t) + \frac{a_k}{m!}(t-t_k)^m, \quad t_k \leq t \leq t_{k+1}$$

where $s^{(j)}(t_k)$, $0\le j\le n-1$ are known from the previous interval $[t_{k-1},t_k]$ and the parameter a_k is determined from the collocation condition

$$s^{(n)}(t_{k+1}) = f(t_{k+1}, s(t_{k+1}), s(g(t_{k+1}))), \quad k=\overline{0,N-1} \tag{9}$$

This method produces a spline function $s\in\mathcal{S}_m$ over the entire interval $[r_j, r_{j+1}]$ with knots $\{t_k\}_{k=0}^{N}$.

Now we show that for h sufficiently small the parameters a_k, $0\le k\le N$ can be uniquely determined from (9).

<u>Theorem 1.</u> <u>If h is sufficiently small then the spline function approximation solution s of the problem (1) defined by the above construction exists and is unique.</u>

Proof. It remains to be proved that a_k can be uniquely determined from (9). Replacing s given by (8) in (9) we obtain:

$$a_k = \frac{(m-n)!}{h^{m-n}} \Big[f\Big(t_{k+1}, A_k(t_{k+1}) + \frac{a_k}{m!}h^m, s(t_k) + \frac{s'(t_k)}{1!}(g(t_{k+1}) - t_k) + \dots + \frac{a_k}{m!}(g(t_{k+1}) - t_k)^m\Big) - A_k^{(n)}(t_{k+1}) \Big] \tag{10}$$

Because $g(t_{k+1}) \le t_{k+1}$ and, f satisfies Lipschitz condition (2), it is easy to see that the right-hand side of (10) is a contraction mapping if

$$h < \big[m(m-1)\dots(m-n+1)/2L\big]^{\frac{1}{n}}$$

and if $|g(t_{k+1}) - t_k| \le h$. If $|g(t_{k+1}) - t_k| > h$ the proof goes similar as in [5].

In order to make a connection between the described spline method and the well known linear multistep methods we present the following theorem.

For any spline function $s\in\mathcal{S}_m$, with equidistant knots $t_k = t_0 + kh$, there is a linear relation between the quantities $s(t_0+kh)$ and $s^{(n)}(t_0+kh)$, $(k=0,1,\dots,m-1)$ given by:

$$\sum_{k=0}^{m-1} a_k^{(m)} s(t_0+kh) = h^n \sum_{k=0}^{m-1} b_k^{(m)} s^{(n)}(t_0+kh) \tag{11}$$

where

$$a_k^{(m)} := (m-1)! \left[Q_{m-1}(k+1) - 2Q_{m-1}(k) + Q_{m-1}(k+1)\right] \tag{12}$$

$$b_k^{(m)} := (m-1)! \, Q_{m+1}(k+1) \tag{13}$$

and

$$Q_{m+1}(t) := \frac{1}{m!} \sum_{j}^{m+1} (-1)^j \binom{m+1}{j} (t-j)_+^m \tag{14}$$

(See more details in [1].

Theorem 2. The values $s(t_k)$ of the spline function defined above construction are exactly the values furnished by the discrete multistep method described by the reccurence relation

$$\sum_{k=0}^{m-1} a_k^{(m)} y_{k-m+j+1} = h^n \sum_{k=0}^{m-1} b^k y_{k-m+j+1}^{(n)}, \quad j=m-1,\ldots,N \tag{15}$$

where coefficients $a_k^{(m)}$ and $b_k^{(m)}$ are given by (11) - (14), if the initial values

$$y_0 = s(t_0),\ y_1 = s(t_1),\ldots,\ y_{m-1} = s(t_{m-1}) \tag{16}$$

are used.

Proof. For

$$h < \left[m(m-1)\ldots(m-n+1) / 2L\right]^{\frac{1}{n}},$$

only one sequence $\{y_j\}$, $j=m-1,\ldots,n$ satisfies the relation (15) with initial values (16). By the consistency relation (11), the sequence $\{s(t_0+jh)\}$, $j=m-1,\ldots,n$ satisfies (15) and obviously has initial values (16). Thus the values $s(t_0+jh)$ must coincide with the values y_j, $j=m-1,\ldots,n$ generated by the corresponding multistep method.

Since $s \in C^{m-1}$, its n^{th} derivative in the knots t_k can be defined by

$$s^{(m)}(t_k) = \tfrac{1}{2}\left[s^{(m)}\left(t_k-\tfrac{h}{2}\right) + s^{(m)}\left(t_k+\tfrac{h}{2}\right)\right], \quad k=\overline{1,N-1}. \tag{17}$$

The purpose now is to discuss the convergence of the spline approximations to the exact solutions as $h \to 0$.

Lemma 1. If

$$|s(t_k) - y(t_k)| \leq Kh^p, \quad |s(g(t_k)) - y(g(t_k))| \leq Kh^p \tag{18}$$

and

$$s^{(n)}(t_k) = f(t_k, s(t_k), s(g(t_k)))$$

Then there exists a constant K_1 such that

$$|s(t_k) - y(t_k)| \leq K_1 h^p \quad \text{and} \quad |s^{(n)}(t_k) - y^{(n)}(t_k)| \leq K_1 h^p$$

Proof. Applying Lipschitz condition (2), it follows that

$$\begin{aligned} |s_*^{(n)}(t_k) - y^{(n)}(t_k)| &= |f(t_k, s(t_k), s(g(t_k))) \\ &\quad - f(t_k, y(t_k), y(g(t_k)))| \\ &\leq L\,[|s(t_k) - y(t_k)| + |s(g(t_k)) - y(g(t_k))|] \\ &\leq 2\,L\,K\,h^p. \end{aligned}$$

Lemma follows by chosing $K_1 = \max\{K, 2LK\}$

Lemma 2 [1]. Let $y \in C^{m+1}[0,T]$ and $s \in \mathcal{S}_m$ be a spline function having its knots at the points $\{t_k\}$ $k=1,\dots,N-1$ and such that the conditions

$$|s^{(r)}(t_k) - y^{(r)}(t_k)| = O(h^{p_r}), \quad |s^{(r)}(g(t_k)) - y^{(r)}(g(t_k))| = O(h^{p_r}), \tag{19}$$

$$r=0,\dots,m-1; \quad k=0,\dots,N-1$$

$$|s^{(m)}(t) - y^{(m)}(t)| = O(h), \quad |s^{(m)}(g(t)) - y^{(m)}(g(t))| = O(h), \tag{20}$$

$t_k \leq t \leq t_{k+1}$ are satisfied. Then

$$|s(t) - y(t)| = O(h^p)$$

where $p := \min\{(r+p_r),\ r=0,\dots,m,\ (p_m=1)\}$ and furthermore

$$|s^{(m)}(t) - y^{(m)}(t)| = O(h), \quad t \in [0,T].$$

Proof is similar as in [1].

3. Numerical example.

Consider the following fourth order initial value problem with deviating argument [2] .

$$y^{(4)}(t) = e^{-t-\sqrt{t}}\sin\sqrt{t} - 9e^{-t}\sin t - e^{-t}y(\sqrt{t}), \quad t \geq 0$$

$$\phi(t) = e^{-t}, \quad t \in [a, 0], \qquad a < 0$$

The discontinuity point is $t_0 = 0$

The exact solution of this problem is

$$y(t) = e^{-t}\sin t \quad , \quad t \geq 0$$

Using the quintic spline approximation for $t \in [0, \frac{1}{5}]$ we obtain

$$s(t) = \phi(0) + \frac{\phi'(0)}{1!}t + \frac{\phi''(0)}{2!}t^2 + \frac{\phi'''(0)}{3!}t^3 + \frac{\phi^{(iv)}(0)}{4!}t^4 + \frac{a_0}{5!}t^5$$

$$= 1 - t + \tfrac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{a_0}{120}t^5$$

From condition (4) it follows

$$a_0 = -8.3889,$$

for $x \in [1/5 , 2/5]$

$$s(t) = \frac{41}{50} - \frac{41}{50}\left(t - \frac{1}{5}\right) + \frac{2}{5}\left(t - \frac{1}{5}\right)^2 - \frac{97}{600}\left(t - \frac{1}{5}\right)^3$$

$$- \frac{17}{600}\left(t - \frac{1}{5}\right)^4 + \frac{a_1}{5}\left(t - \frac{1}{5}\right)^5$$

From equation (5) it follows

$$a_1 = \frac{417}{500} \quad .$$ And so on.

References

1. Loscalzo, F.R. and Talbot, T.D., Spline functions approximations for solutions of ordinary differential equations, SIAM J. Num. Anal., 4(1967), 433 - 445.

2. Lu-San Chen and Chen-Chih Yeh, Necessary and sufficient conditions for asymptotic decay of oscillations in delayed functional equations, Proc. Royal Soc. of Edinburgh 91A (1981), 135-145.

3. Micula, G., Numerical integration of differential equation $y^{(n)}=f(x,y)$ by spline functions, Revue D'anal. Numer. Mathematica 17, 9(1972), 1385-1389.

4. Micula, G. and Akça, H. Approximate solutions of the second order differential equations with deviating argument by spline functions, Mathematica 30(53), 1(1988), 37-46.

5. Rus, A.I., Principii şi aplicatii ale teoriei punctului fix, Ed. Dacia, Cluj-Napoca, 1979.

Faculty of Art and Sciences
Erciyes University
Kayseri, Turkey

Faculty of Mathematics
University of Cluj-Napoca
Romania