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ON THE CHORD METHOD

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In the paper /1/ I.K.Argyros considers as divided difference of the mapping $f: X_1 \to X_2$, where X_1 and X_2 are Banach spaces, a linear mapping $[x,y;f] \in \mathcal{L}(X_1,X_2)$ which fulfils the following conditions:

(a) $[x,y;f](y-x) = f(y) - f(x)$ for every $x,y \in D$, where $D \subseteq X_1$ is a subset of X_1;

(b) there exist the real constants $l_1 \geq 0$, $l_2 \geq 0$, $l_3 \geq 0$ and $p \in (0,1]$ such that for every $x,y,u \in D$ the following inequality holds:

$$\|[y,u;f] - [x,y;f]\| \leq l_1 \|x-u\|^p + l_2 \|x-y\|^p + l_3 \|y-u\|^p .$$

In /1/ the hypothesis that the equation:

(1) $f(x) = 0$

admits a simple solution $x^* \in D_0 \subset D$ is adopted, and conditions for the convergency of the sequence $(x_n)_{n \geq 0}$ generated by the chord method:

(2) $x_{n+1} = x_n - [x_{n-1},x_n;f]^{-1} f(x_n), \quad x_0,x_1 \in D_0, \quad n = 1,2,\ldots$

are given.

In a recent paper /2/ there is shown that, with the hypotheses considered in /1/, the convergency speed of the sequence generated by (2) and the error estimation are featured by the inequality:

(3) $$\|x^* - x_{n+1}\| \leq \alpha \, d_0^{t_1^{n+1}} ,$$

where α is a precised constant, $0 < d_0 < 1$, and t_1 is the positive root of the equation $t^2-t-p = 0$.

We shall admit further down that the divided difference operator fulfils the conditions (a) and (b), and search for supplementary conditions in order to make equation (1) admit a solution x^* into a precised domain D_0 and the sequence $(x_n)_{n \geq 0}$ generated by (2) converge to this solution.

Observe firstly that the identity:

(4) $x_n - [x_{n-1},x_n;f]^{-1} f(x_n) = x_{n-1} - [x_{n-1},x_n;f]^{-1} f(x_{n-1})$

holds for every $n = 1,2,\dots,$ with the hypothesis that the linear mappings $[x_{n-1},x_n;f]$ admit inverse mappings.

The following identity also holds:

(5) $f(x_{n+1}) = f(x_n) + [x_{n-1},x_n;f](x_{n+1} - x_n) +$

$+ ([x_n,x_{n+1};f] - [x_{n-1},x_n;f])(x_{n+1} - x_n), \quad n = 1,2,\dots$

Let $\alpha > 0$, $B > 0$, $d_o \in (0,1)$, and $x_1, x_o \in X_1$. Consider the sphere:

(6) $$U = \left\{ x \in X_1 \;\middle|\; \|x-x_o\| \le \frac{B\alpha d_o}{1-d_o^{t_1-1}} \right\},$$

where $t_1 = \dfrac{1+\sqrt{1+4p}}{2}$, that is, the positive root of the equation:

(7) $t^2 - t - p = 0$.

The following theorem holds:

THEOREM. If the divided difference $[x,y;f]$ fulfils the conditions (a) and (b) for every $x,y \in U$, and the following hypotheses:

(i) the mapping $[x,y;f]$ admits a bounded inverse mapping for every $x,y \in U$, namely there exists a constant $B > 0$ such that $\|[x,y;f]^{-1}\| \le B$

(ii) $\alpha = \dfrac{1}{B^{(1+p)/p}(l_1+l_2+l_3)^{1/p}}$;

(iii) $\|x_1-x_o\| \le B\alpha d_o$, $\|f(x_o)\| \le \alpha d_o$, $\|f(x_1)\| \le \alpha d_o^{t_1}$

are also fulfilled, then equation (1) has at least one solution x^* Um and the sequence $(x_n)_{n \ge 0}$ generated by (2) converges to x^*, the convergence speed and the error estimation being featured by the inequality:

$$\| x^{*}-x_n \| \leq \frac{B\alpha d_o^{t_1^n}}{1 - d_o^{t_1^n (t_1-1)}} .$$

Proif. From (2) for n=1 we deduce:

$$\| x_2-x_1 \| \leq B \cdot \| f(x_1) \| \leq B\alpha d_o^{t_1} ,$$

from which, taking also into account (iii), follows:

$$\| x_2-x_o \| \leq \| x_2-x_1 \| + \| x_1-x_o \| \leq B\alpha d_o^{t_1} + B\alpha d_o =$$

$$= B\alpha d_o (1+d_o^{t_1-1}) < \frac{B\alpha d_o}{1-d_o^{t_1-1}} ,$$

from which it results that $x_2 \in U$.

Using the fact that $x_2 \in U$, the identities (4) and (5), and the inequality (b), we obtain:

$$\| f(x_2) \| \leq \| x_2-x_1 \| (l_1 \| x_2-x_o \|^p + l_2 \| x_1-x_o \|^p + l_3 \| x_2-x_1 \|^p) \leq$$

$$B \| f(x_1) \| (l_1 B^p \| f(x_o) \|^p + l_2 \| x_1-x_o \|^p + l_3 B^p \| f(x_1) \|^p) \leq$$

$$B\alpha d_o^{t_1} (l_1 B^p \alpha^p d_o^p + l_2 B^p \alpha^p d_o^p + l_3 B^p \alpha^p d_o^{p t_1}) \leq$$

$$B^{p+1} \alpha^{p+1} d_o^{t_1+p} (l_1+l_2+l_3 d_o^{p(t_1-1)}) =$$

$$= B^{p+1} \alpha^{p+1} (l_1+l_2+l_3 d_o^{p(t_1-1)}) d_o^{t_1^2-p} \leq \alpha d_o^{t_1^2} ,$$

since $\alpha^p B^{p+1} (l_1+l_2+l_3 d_o^{p(t_1-1)}) \leq \alpha^p B^{p+1} (l_1+l_2+l_3) < 1$.

From the above inequality follows therefore:

$$\| f(x_2) \| \leq \alpha d_o^{t_1^2} .$$

Suppose by induction that:

(a') $x_i \in U$, $i = 0,1,\dots,k$;

(b') $\|f(x_i)\| \leq \alpha d_0^{t_1^i}$, $i = 1,2,\dots,k$.

Then, for x_{k+1} we have:

$$\|x_{k+1} - x_0\| \leq \|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + \dots + \|x_1 - x_0\| \leq$$

$$B\|f(x_k)\| + B\|f(x_{k-1})\| + \dots + B\alpha d_0 \leq$$

$$B\alpha d_0^{t_1^k} + B\alpha d_0^{t_1^{k-1}} + \dots + B\alpha d_0 =$$

$$B\alpha d_0 (1 + d_0^{t_1-1} + d_0^{t_1^2-1} + \dots + d_0^{t_1^k-1}) \leq$$

$$B\alpha d_0 (1 + d_0^{t_1-1} + d_0^{2(t_1-1)} + \dots + d_0^{k(t_1-1)}) < \frac{B\alpha d_0}{1 - d_0^{t_1-1}}$$

from which follows that $x_{k+1} \in U$. Proceeding now for x_{k+1} as in the case of x_2, we obtain:

$$\|f(x_{k+1})\| \; B^{p+1}\alpha^{p+1} (l_1 + l_2 + l_3 d_0^{p t_1^{k-1} \cdot (t_1-1)}) . d_0^{t_1^{k-1}(t_1+p)} \leq$$

$$B^{p+1}\alpha^{p+1} (l_1 + l_2 + l_3)\, d_0^{t_1^{k+1}} \leq \alpha d_0^{t_1^{k+1}} .$$

It results therefore that the relations (a') and (b') hold for every $i \in \mathbb{N}$.

Now we shall show that the sequence $(x_n)_{n \geq 0}$ is fundamental. Indeed, for every $n, s \in \mathbb{N}$ we have:

$$\|x_{n+s} - x_n\| \leq \sum_{k=n}^{n+s-1} \|x_{k+1} - x_k\| \leq \sum_{k=n}^{n+s-1} B\|f(x_k)\| \leq B\alpha \sum_{k=n}^{n+s-1} d_0^{t_1^k} =$$

$$= B\alpha d_0^{t_1^n} \sum_{k=n}^{n+s-1} d_0^{t_1^k - t_1^n} = B\alpha d_0^{t_1^n} \sum_{k=n}^{n+s-1} d_0^{t_1^n (t_1^{k-n} - 1)} \leq$$

$$B\alpha d_0^{t_1^n} \sum_{k=n}^{n+s-1} d_0^{t_1^n (k-n)(t_1-1)} = B\alpha d_0^{t_1^n} \sum_{k=n}^{n+s-1} \left(d_0^{t_1^n (t_1-1)}\right)^{k-n} \le$$

$$\le \frac{B\alpha d_0^{t_1^n}}{1-d_0^{t_1^n (t_1-1)}} .$$

By the last inequality and the fact that $0 < d_0 < 1$ and $t_1 > 1$ follows that the sequence $(x_n)_{n \geq 2}$ is fundamental. For $s \to \infty$, from the inequality:

$$\| x_{n+s} - x_n \| \le \frac{B\alpha d_0^{t_1^n}}{1-d_0^{t_1^n (t_1-1)}}$$

follows the inequality:

$$\| x^* - x_n \| \le \frac{B\alpha d_0^{t_1^n}}{1-d_0^{t_1^n (t_1-1)}} .$$

In /1/ Argyros showed that if the divided difference $[x,y;f]$ fulfils the conditions (a) and (b) then f is Fréchet differentiable and $[x,x;f] = f'(x)$. From this fact follows that the mapping f is continuous on U; hence at limit for $n \to \infty$ in the inequality:

$$\| f(x_n) \| \le \alpha d_0^{t_1^n} ,$$

one obtains:

$$\| f(x^*) \| \le 0 ,$$

from which results $f(x^*) = 0$. With this the theorem is entirely proved.

Remark. In /5,6/ Schmidt imposes to the divided difference conditions similar to the conditions (a) and (b) given by Argyros in /1/, but for $p = 1$. The same conditions are reproduced in /2/, too.

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