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NUMERICAL METHOD FOR A CLASS OF SINGULAR INITIAL VALUE PROBLEMS

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ABSTRACT.

For the numerical integration of initial value problems (1.1) some explicit single step methods of the form (2.1) - (2.2) and (4.20)-(4.21) having the order of accuracy $O(h^5)$ for the solution $y(x)$ and $O(h^4)$ for its derivative $y'(x)$, are presented.

1. Introduction

We consider the singular Cauchy's problems

$$y'' + \frac{2}{x} y' = Ay + f(x), \quad y(0) = y_0, \; y'(0) = y'_0 \tag{1.1}$$

where A is a real constant and $f \in C[0,1]$.

These types of problems and analogous two-point boundary value problemes arise in many physical problems and they were investigated by several authors: Chawla and Katti [2], Jain and Jain [5], when the right-hand side of (1.1) is nonlinear. Russel and Shampine [6] have shown that the problem (1.1) has a unique solution if $f \in C[0,1]$ and $A < \pi^2$. Everywhere in this paper we suppose that these conditions are satisfied and moreover that there exist $f'(x)$ and $f''(x)$ over $[0,1]$.

In the present paper we have derived a numerical method for the problems (1.1), which is analogous to explicit Runge-Kutta type methods, in a similar way like in [4] and [5].

2. The construction of the numerical method

Let $0 = x_0 < x_1 < x_2 < \cdots < x_N = 1$, be a set of points such that $x_{n+1} - x_n = h$; $n = o,1,2,\ldots,N-1$ and let $a_i, i = 1(1)4$; be real numbers, $0 < a_i < 1$, $a_1 = 0$. We denote by y_n and y'_n the aproximate values of the exact solution $y(x_n)$ and of its derivative $y'(x_n)$, respectively, of the problem (1,1). We note that for each value of $n=o,1,2,\ldots,N-1$, tho approximation to $y(x_{n+1})$ and to $y'(x_{n+1})$ is computed as though y_n and y'_n were exactely equal to $y(x_n)$ and $y'(x_n)$ respectively.

We define the approximate value to $y(x_{n+1})$ and to $y'(x_{n+1})$ in the form:

$$y_{n+1} = y_n + \frac{x_n}{x_{n+1}} h y'_n + h^2 \sum_{i=1}^{4} e_i(x_n+a_i h)k_i \tag{2.1}$$

$$y'_{n+1} = \left(\frac{x_n}{x_{n+1}} \right)^2 y'_n + h \sum_{i=1}^{4} e'_i(x_n+a_i h)k_i \tag{2.2}$$

with

$$k_i = Ay(x_n+a_i h) + f(x_n+a_i h); \quad i = 1,2,3,4, \tag{2.3}$$

and e_i, e'_i ; $i = 1,2,3,4$ are real constants.

If $y(x)$ is the exact solution of (1.1) then we can take

$$k_i = y''(x_n + a_i h) + \frac{2}{x_n+a_i h} y'(x_n+a_i h) \tag{2.4}$$

The formulas (2.1)-(2.2) mean a four stage numerical single step method for the problem (1.1). In order to specify the order or accuracy of the method (2.1)-(2.2) we will expand. y_{n+1}, y'_{n+1}, from (2.1), (2.2) and the exact values $y(x_{n+1})$, $y'(x_{n+1})$ by Tayler expansions in powers of h. After a tedious computation, and rearranging the terms we obtain

$$y_{n+1} = y_n + \left(h \frac{x_n}{x_{n+1}} + 2h^2 \sum_{i=1}^{4} e_i\right) y'_n + \left(h^2 x_n \sum_{i=1}^{4} e_i + 3h^3 \sum_{i=2}^{4} e_i a_i\right) y''_n +$$

$$+\left(h^3 x_n \sum_{i=2}^{4} c_i a_i + 2h^4 \sum_{i=2}^{4} c_i a_i^2\right) y_n^{(3)} + \left(\frac{1}{2} h^4 x_n \sum_{i=2}^{4} c_i a_i^2 + \right.$$

$$\left. + \frac{5}{6} h^5 \sum_{i=2}^{4} c_i a_i^3\right) y_n^{(4)} + \left(\frac{1}{6} h^5 x_n \sum_{i=2}^{4} c_i a_i^3 + \frac{1}{4} h^6 \sum_{i=2}^{4} c_i a_i^4\right) y_n^{(5)} +$$

$$+ O(h^7), \tag{2.5}$$

$$y'_{n+1} = \left(\frac{x_n^2}{x_{n+1}^2} + 2h \sum_{i=1}^{4} c'_i\right) y'_n + \left(h\, x_n \sum_{i=1}^{4} c'_i + 3h^2 \sum_{i=2}^{4} c'_i a_i\right) y''_n +$$

$$+ \left(h^2 x_n \sum_{i=2}^{4} c'_i a_i + 2h^3 \sum_{i=2}^{4} c'_i a_i^2\right) y_n^{(3)} + \left(\frac{1}{2} h^3 x_n \sum_{i=2}^{4} c'_i a_i^2 + \right.$$

$$\left. + \frac{5}{6} h^4 \sum_{i=2}^{4} c'_i a_i^3\right) y_n^{(4)} + \left(\frac{1}{6} h^4 x_n \sum_{i=2}^{4} c'_i a_i^3 + \frac{1}{4} h^5 \sum_{i=2}^{4} c'_i a_i^4\right) y_n^{(5)} +$$

$$+ \left(\frac{1}{24} h^5 x_n \sum_{i=2}^{4} c'_i a_i^4 + \frac{7}{120} h^6 \sum_{i=2}^{4} c'_i a_i^5\right) y_n^{(6)} + \left(\frac{1}{120} h^6 x_n \sum_{i=2}^{4} c'_i a_i^5\right) y_n^{(7)} +$$

$$+ O(h^7), \tag{2.6}$$

$$y(x_{n+1}) = y_n + h y'_n + \frac{h^2}{2} y''_n + \frac{h^3}{6} y_n^{(3)} + \frac{h^4}{24} y_n^{(4)} + y_n^{(4)} +$$

$$+ \frac{h^5}{120} y_n^{(5)} + \frac{1}{720} h^6 y_n^{(6)} + O(h^7), \tag{2.7}$$

$$y'(x_{n+1}) = y'_n + h y''_n + \frac{h^2}{2} y_n^{(3)} + \frac{h^3}{6} y_n^{(4)} + \frac{h^4}{24} y_n^{(5)} +$$

$$+ \frac{h^5}{120} y_n^{(6)} + \frac{h^6}{720} y_n^{(7)} + O(h^7). \tag{2.8}$$

Identifying the coefficients of $y_n^{(i)}$; $i = 1,2,3,4$ from (2.5), (2.7) and from (2.6),(2.8) and simplifying we can find the following algebraic system for a_i, c_i, c'_i.

$$\sum_{i=1}^{4} c_i = \frac{1}{2x_{n+1}}, \qquad \sum_{i=1}^{4} c'_i = \frac{x_n + x_{n+1}}{2x_{n+1}^2},$$

$$\sum_{i=2}^{4} c_i a_i = \frac{1}{6x_{n+1}}, \qquad \sum_{i=2}^{4} c'_i a_i = \frac{x_n + 2x_{n+1}}{6x_{n+1}^2},$$

$$\sum_{i=2}^{4} c_i a_i^2 = \frac{1}{12x_{n+1}}, \qquad \sum_{i=2}^{4} c'_i a_i^2 = \frac{x_n + 3x_{n+1}}{12\, x_{n+1}^2} \tag{2.9}$$

$$\sum_{i=2}^{4} e_i a_i^3 = \frac{1}{2o x_{n+1}} , \qquad \sum_{i=2}^{4} e_i' a_i^3 = \frac{x_n + 4x_{n+1}}{2o\, x_{n+1}^2} .$$

3. **The truncation error of the method**

We denote by

$$T_n = y(x_{n+1}) - y_{n+1} , \quad T_n' = y'(x_{n+1}) - y_{n+1}'$$

the truncation error when the method (2.1)-(2.2) is applied to problem (1.1). After computation from (2.5)-(2.8) we find

$$T_n = \frac{h^6}{72o} A_6 + O(h^7) , \tag{3.1}$$

$$T_n' = \frac{h^5}{12o} A_5' + \frac{h^6}{72o} A_6' + O(h^7) , \tag{3.2}$$

where

$$A_6 = y_n^{(5)}\left(\frac{6}{x_{n+1}} - 18o \sum_{i=2}^{4} e_i a_i^4\right) + y_n^{(6)}\left(1 - 30\, x_n \sum_{i=2}^{4} e_i a_i^4\right) , \tag{3.3}$$

$$A_5' = y_n^{(5)}\left(\frac{x_n + 5x_{n+1}}{x_{n+1}^2} - 30 \sum_{i=2}^{4} e_i' a_i^4\right) + y_n^{(6)}\left(1 - 5x_n \sum_{i=2}^{4} e_i' a_i^4\right) , \tag{3.4}$$

$$A_6' = y_n^{(7)}\left(1 - 6\, x_n \sum_{i=2}^{4} e_i' a_i^5\right) - 42\, y_n^{(6)} \sum_{i=2}^{4} e_i' a_i^5 \tag{3.5}$$

4. **The affective use of the method**

When the problem (1.1) is solved numerically with the method (2.1)-(2.2) we must know the constants a_i, e_i, e_i' ; $i = 1(1)4$ and whe must compute k_i, $i = 1(1)\,4$ at every step. But

$$k_i = A\, y(x_n + a_i h) + f(x_n + a_i h) = A y_{ni} + f_{ni} , \tag{4.1}$$

and we do not know the exact solution $y(x)$. To remove this we will replace k_i in (2.1)-(2.2) with $\bar{k}_i$, $i = 1(1)4$, where

$$\bar{k}_i = A\, \bar{y}_{ni} + f_{ni} \; ; \; i = 1(1)4 , \tag{4.2}$$

and $\bar{y}_{ni}$ are the approximate values of $y_{ni} = y(x_n + a_i h)$

defined as

$$\bar{y}_{ni} = y_n + a_i(y_n - y_{n-1}) + h^2 \sum_{j=1}^{i-1} b_{ij} \bar{k}_j \, , \qquad (4.3)$$

Here b_{ij} are real constants. For $\bar{y}_{ni}$ and $\bar{k}_i$, $i = 1(1)4$ we get

$$\bar{y}_{n1} = y_n = y_{n1}, \quad \bar{k}_1 = Ay_{n1} + f_{n1} = k_1 \, , \qquad (4.4)$$

$$\bar{y}_{n2} = y_{n2} + h^2 \left[b_{21}(Ay_n + f_{n1}) - \frac{1}{2} a_2(1+a_2) \, y''_n \right] +$$

$$+ \frac{h^3}{6} a_2(1 - a_2^2) \, y_n^{(3)} + O(h^4) \, . \qquad (4.5)$$

Equating the coefficient of h^2 to zero we get

$$b_{21} = \frac{1}{2} a_2(1+a_2) - a_2(1+a_2) \frac{y'_n}{x_n(Ay_n + f_{n1})} \, , \qquad (4.6)$$

$$\bar{y}_{n2} = y_{n2} + \frac{h^3}{6} a_2(1-a_2^2) \, y_n^{(3)} - \frac{h^4}{24} a_2(1-a_2^3) \, y_n^{(4)} + O(h^5) \qquad (4.7)$$

$$\bar{k}_2 = A\bar{y}_{n2} + f_{n2} = k_2 + \frac{h^3}{6} Aa_2(1-a_2^2) \, y_n^{(3)} + O(h^4) \qquad (4.8)$$

In a similar way we obtain

$$\bar{y}_{n3} = y_{n3} + h^4\left(\frac{b_{32} \, a_2^2}{2} Ay_n^{(2)} - \frac{1}{24} a_3(1+a_3^3) y_n^{(4)}\right) + O(h^5) \qquad (4.9)$$

$$\bar{k}_3 = k_3 + \frac{A}{2} h^4\left(b_{32} a_2^2 \, A \, y_n^{(2)} - \frac{a_3(1+a_3^3)}{12} \, y_n^{(4)}\right) + O(h^5) \qquad (4.10)$$

$$\bar{y}_{n4} = y_{n4} + O(h^5) \qquad (4.11)$$

$$\bar{k}_4 = k_4 + O(h^5) \qquad (4.12)$$

For the coefficients bij we have

$$b_{31} = \frac{1}{2} a_3(1+a_3) - a_3(1+a_3) \frac{y'_n}{x_n(Ay_n + f_{n1})} - \qquad (4.13)$$

$$- \frac{a_3(1-a_3^2)}{a_2 A y'_n} \cdot \frac{Ay_n + f_{n2}}{Ay_n + f_{n1}} \left[\frac{2A \, y'_n}{x_n^2} + \left(A - \frac{4}{x_n^3}\right) y'_n + \frac{2f_{n1}}{x_n^2} + f'_{n1} \right]$$

$$b_{32} = \frac{a_3(a_3^2 - 1)}{a_2 A y'_n} \left(\frac{2A \, y_n}{x_n^2} + \left(A - \frac{4}{x_n^3}\right) y'_n + \frac{2f_{n1}}{x_n^2} + f'_{n1} \right] , \qquad (4.15)$$

$$b_{42}= \frac{a_4}{12\, a_2(a_2-a_3)}\,(E_n - a_3 F_n)\ , \qquad (4.16)$$

$$b_{43}= \frac{a_4}{12\, a_3 A(a_3-a_2)}\,(E_n - a_2\, F_n)\ , \qquad (4.17)$$

$$b_{41}= \frac{1}{Ay_n+f_{n1}}\Big[\tfrac{1}{2}a_4(1+a_4)(A\, y_n+f_{n1}-\frac{2}{x_n}\, y'_n -$$

$$- b_{42}(A\, y_n+ f_{n2})- b_{43}(Ay_n+ f_{n3})\Big] \qquad (4.18)$$

where $f'_{n1} = f'(x_n+a_1h) = f'(x_n)$, $f''_{n1}= f''(x_n)$, and

$$E_n = \frac{(a_4^3+1)\, x_n}{A\, x_n y_n+x_n f_{n1}-2y'_n}\Big[\,(A^2-\frac{8A}{x_n^3})y_n-2(\frac{A}{x_n}\,\frac{4}{x_n^2}-\frac{10}{x_n^4})\, y'_n\Big],$$

$$F_n= 2(a_4^3-1)\Big[\frac{2Ay_n}{x_n^2 y'_n}+(A-\frac{4}{x_n^3})+\frac{2f_{n1}}{x_n^2 y'_n}+\frac{f'_{n1}}{y'_n}\Big], \qquad (4.19)$$

$n = 1,\ 2,\ \ldots,\ N-1$

Thus we have

<u>THEOREM 4.1</u>

The numerical method defined by

$$y_{n+1}= y_n + \frac{x_n}{x_{n+1}}\, hy'_n + h^2\sum_{i=1}^{4} e_i(x_n+a_ih)\, E_i\ , \qquad (4.2o)$$

$$y'_{n+1} = (\frac{x_n}{x_{n+1}})^2\, y'_n+ h\sum_{i=1}^{4} e'_i\,(x_n+a_ih)\, E_i\ , \qquad (4.21)$$

for the problem (1.1) with e_i, a_i, e'_i satisfying the system (2.9) and E_i given by (4.4), (4.8), (4.1o), (4.12) provide aproximate values to $y(x)$ of order four and for $y'(x)$ of order three.

<u>Proof</u>

Using the relations (4.4),(4.8), (4.1o),(4.12) in (4.2o) and (4.21) we have

$$Y_{n+1} = y_n + \frac{x_n}{x_{n+1}} h y'_n + h^2 \sum_{i=1}^{4} e_i(x_n + a_i h) k_i + O(h^5), \tag{4.23}$$

$$Y'_{n+1} = \left(\frac{x_n}{x_{n+1}}\right)^2 y'_n + h \sum_{i=1}^{4} e'_i(x_n + a_i h)\, k_i + O(h^4), \tag{4.24}$$

and taking account of (2.1), (2.2)

$$Y_{n+1} = y_{n+1} + O(h^5), \quad Y'_{n+1} = y'_{n+1} + O(h^4). \tag{4.25}$$

Now, if e_i, e'_i, a_i, $i=1(1)4$ satisfies (2.9) then from (3.1), (3.2) we obtain

$$y_{n+1} = y(x_{n+1}) + O(h^6), \quad y'_{n+1} = y'(x_{n+1}) + O(h^5)$$

and

$$Y_{n+1} - y(x_{n+1}) = O(h^5), \quad Y'_{n+1} - y'(x_{n+1}) = O(h^4)$$

These relations prove the statement.

REMARK 4.1.

The method (2.1)-(2.2) and respectively (4.20)-(4.21) can be applied for solving numerically the singular boundary value problems of the type

$$y'' + \frac{2}{x} y' = Ay + f(x), \quad y(0) = a, \; y(1) = b, \tag{4.26}$$

if we use the "Shooting method", [5] . This method is a technique which adjusts the initial data so that the problem (4.26) becomes ar initial value problem of the type (1.1).

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