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CONVERGENCE PROPERTIES OF SOME APPROXIMATION PROCEDURES

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1. Introduction. The divergence phenomenon of some usual appoximation procedures for real functions of a real variable, such as the development in Fourier trigonometric series, the Lagrange interpolation and the quadrature formulas, has been made evident ever since the past century and the first three decades of our century. The subsequent researches pointed out some dramatic aspects of this phenomenon: the almost everywhere divergence, the superdense divergence or the divergence on the whole definition interval of the singular functions for the considered procedure, and these singular functions constitute themselves superdense sets in the space of continuous functions. To attenuate the effect of this phenomenon some modification of approximation procedures have been advanced consisting in the formation of Cesàro or integral means.

Following the last idea, in Sections 3 and 4 of this note we establish the convergence in measure of Fourier trigonometric series and of some types of Lagrange interpolation. The principle of the condensation of singularities is used in Section 5 to characterize those simple quadrature formulas which are unboundedly divergent on superdense sets in the space of continuous functions.

2. Principles of the condensation of singularities. A subset S of a topological space T is said to be superdense in T if S is uncountable dense and residual in T (i. e., S is the complement of a set of first category). From the Baire's theorem it follows that a subset S of a complete metric space T is superdense in T if

and only if S is uncountable and residual. Given two normed spaces X and Y over the same field K (which is $\mathbb{R}$ or $\mathbb{C}$), denote by $(X,Y)^*$ the vector space over K of all linear and continuous applications $A: X \longrightarrow Y$ endowed with the uniform norm $\|A\| = \sup\{\|A(x)\|: x \in X, \|x\| \leq 1\}$. With each family $\mathcal{A}$ in $(X,Y)^*$ we associate the <u>set of singularities</u> of $\mathcal{A}$:

$$S_{\mathcal{A}} = \{x \in X: \sup\{\|A(x)\|: A \in \mathcal{A}\} = \infty\} .$$

The following two theorems contain the well-known principles of the condensation and of the double condensation of singularities, respectively see 2 , Theorems 3.1, (iv), and 5.2):

2.1. THEOREM. <u>If</u> X <u>is a Banach space</u>, Y <u>is a normed space and</u> $\mathcal{A} \subset (X,Y)^*$ <u>is a uniformly unbounded family, i. e.</u>, $\sup\{\|A\|: A \in \mathcal{A}\} = \infty$, <u>then the set</u> $S_{\mathcal{A}}$ <u>is superdense in</u> X.

2.2. THEOREM. <u>Let</u> X <u>be a Banach space</u>, Y <u>be a normed space</u>, <u>and</u> T <u>be a separable complete metric space without isolate points</u>. <u>Suppose</u> $\mathcal{A} = \{A_i: i \in I\}$ <u>is a family of mappings</u> $A_i: X \times T \longrightarrow Y$ <u>possessing the following properties</u>:

1. $A_i(\cdot,t) \in (X,Y)^*$ <u>for each</u> $i \in I$ <u>and each</u> $t \in T$;
2. $A_i(x,\cdot): T \longrightarrow Y$ <u>is continuous for each</u> $x \in X$ <u>and each</u> $i \in I$;
3. <u>there is a subset</u> T_o <u>of</u> T <u>with</u> $\overline{T}_o = T$ <u>and, for each</u> $t \in T_o$, $\sup\{\|A_i(\cdot,t)\|: i \in I\} = \infty$.

<u>Then there exists a superdense subset</u> X_o <u>of</u> X <u>such that the set</u> $\{t \in T: \sup\{\|A_i(x,t)\|: i \in I\} = \infty\}$ <u>is superdense in</u> T <u>for each</u> $x \in X_o$.

3. <u>Convergence properties of Fourier trigonometric series</u>. Denote by T the compact interval $[0,1]$ and consider the trigonometric system $\{e_k: k \in \mathbb{Z}\}$, where $e_k: T \longrightarrow \mathbb{C}$ is the function defined by $e_k(t) = e^{2\pi ikt}$. With each $x \in L^1(T)$ we associate the Fourier coefficients $c_k = (x|e_k)$, the sequence of partial sums

$$s_n(x)(t) = \sum_{k=-n}^{n} c_k e_k(t),$$

and the Fourier trigonometric series $\sum_{k\in Z} c_k e_k$. Clearly, if x is a trigonometric polynomial, then $s_n(x) = x$ for all sufficiently large n, so that it is natural to pose the following question: for what functions x in $L^1(T)$, points t in T, and kinds of convergence we have

(1) $$s_n(x)(t) \longrightarrow x(t) \quad \text{when} \quad n \longrightarrow \infty \ ?$$

P. du Bois-Reymond (1876) constructed, for each $t \in T$, the first example of function in $C(T)$ for which (1) fails. Subsequently, more such example had been appearing. A. N. Kolmogoroff (1926) constructed a function $x \in L^1(T)$ such that $s_n(x)(t) \not\longrightarrow x(t)$ at each $t \in T$. Using Theorem 2.2, the following result on double condensation of singularities for Fourier trigonometric series holds: There is a superdense subset X_o of $C(T)$ such that for each $x \in X_o$ the set $\{t \in T:\ \sup\{|s_n(x)(t)| : n \in N\} = \infty\}$ is superdense in T (see [2], Theorem 6.1). Nevertheless, some positive results hold: 1° For each $x \in L^2(T)$, we have $s_n(x) \longrightarrow x$ in quadratic integral mean, i. e., $\int_T |s_n(x)(t) - x(t)|^2 dt \longrightarrow 0$ as $n \longrightarrow \infty$ (F. Riesz and E. Fischer, 1907); 2° For each $x \in L^2(T)$, we have $s_n(x)(t) \longrightarrow x(t)$ for almost all t in T (L. Carleson, 1966); and also

3.1. THEOREM (A. N. Kolmogoroff [5]). <u>If</u> $0 < p < 1$, <u>then for each</u> $x \in L^1(T)$ <u>we have</u> $\int_T |s_n(x)(t) - x(t)|^p dt \longrightarrow 0$ <u>as</u> $n \longrightarrow \infty$.

From this we derive the convergence in measure of Fourier trigonometric series:

3.2. THEOREM. <u>For each</u> $x \in L^1(T)$ <u>we have</u> $s_n(x) \longrightarrow x$ <u>in measure on</u> T <u>as</u> $n \longrightarrow \infty$.

<u>Proof</u>. We must prove that for each $\sigma > 0$ one has $\mu(E_{\sigma,n}) \longrightarrow 0$ as $n \longrightarrow \infty$, where $E_{\sigma,n} = \{t \in T:\ |s_n(x)(t) - x(t)| \geq \sigma\}$ and μ is the Lebesgue measure. Let $\varepsilon > 0$. From Theorem 3.1 with a fixed $p \in\ ]0,1[$, it follows that there is an $n_o \in N$ such that

$$\int_T |s_n(x)(t) - x(t)|^p dt < \varepsilon \sigma^p \quad \text{for all } n \geq n_0.$$

Then $\mu(E_{\sigma,n}) < \varepsilon$ for all $n \geq n_0$ since

$$\sigma^p \cdot \mu(E_{\sigma,n}) = \int_{E_{\sigma,n}} \sigma^p dt \leq \int_{E_{\sigma,n}} |s_n(x)(t) - x(t)|^p dt < \varepsilon \sigma^p. \blacksquare$$

Remark that Theorem 3.2 is not more valid for multiple Fourier trigonometric series see [6] .

4. Convergence properties of interpolation procedures. Denote by T an interval in $\mathbb{R}$ and by $\mathfrak{M}$ an infinite triangular matrix of nodes in T: $\mathfrak{M} = \{(t_n^1, \ldots, t_n^n): t_n^1 < \ldots < t_n^n, n \in \mathbb{N}\}$. With the nodes $t_n^1, \ldots, t_n^n$ of the n-th row in $\mathfrak{M}$, and with a function x: $T \longrightarrow \mathbb{K}$ we associate the Lagrange interpolation polynomial $L_n(x) = L_n^{\mathfrak{M}}(x)$ satisfying

(2) $\quad L_n(x)(t_n^i) = x(t_n^i)$, $i \in \{1, \ldots, n\}$, and $L_n(x) = x$ for each polynomoal x of degree $\leq n - 1$.

The plynomial $L_n(x)$ is given by

(3) $$L_n(x)(t) = \sum_{i=1}^n x(t_n^i) \ell_n^i(t),\ \ell_n^i(t) = \frac{\omega_n(t)}{(t-t_n^i)\omega_n'(t_n^i)},\ \omega_n(t) = \prod_{i=1}^n (t-t_n^i).$$

From (2) we derive that, for each polynomial x, $L_n(x) = x$ whenever n is sufficiently large. Therefore, it is natural to ask: For what matrices $\mathfrak{M}$ of nodes, what functions x and points t, and what type of convergence we have

(4) $$L_n^{\mathfrak{M}}(x)(t) \longrightarrow x(t) \text{ as } n \longrightarrow \infty \ ?$$

Ch. Méray (1896) and C. Runge (1901) constructed the first examples of functions in $C(T)$, with T compact and with equidistant nodes for which (4) fails at some points. P. Erdös and P. Vértesi (1980) showed that for any matrix $\mathfrak{M}$ of nodes in $T = [-1,1]$ there is a measurable subset E of T, with $\mu(E) = 2$, and there exists $x \in C(T)$ such that $\sup\{|L_n(x)(t)|: n \in \mathbb{N}\} = \infty$ for each $t \in E$. In [2], Theorem 8.1, is proved that for any matrix $\mathfrak{M}$ of nodes in T there is a superdense subset X_0 of $C(T)$ such that the set

$\{t \in T: \sup\{|L_n(x)(t)| : n \in N\} = \infty\}$ is superdense in T for each x in X_0. However, if $T = [-1,1]$ and $(\omega_n)_{n\in N}$ is the sequence of orthogonal polynomials with respect to a weight $w: T \longrightarrow \mathbb{R}$ (i. e., w is a nonnegative function in $L^1(T)$ with $\int_T w(t)dt > 0$), and the nodes on the n-th row of $\mathfrak{M}$ are the zeros $t_n^1 < \dots < t_n^n$ of the polynomial ω_n, then the following positive result holds:

4.1. THEOREM (P. Erdős and P. Turán [3]). _If_ $w(t) \geq w_0 > 0$ _for all_ $t \in T$, _then for each_ $x \in C(T)$ _we have_ $L_n(x) \longrightarrow x$ _in_ $L^2(T)$, _i. e.,_ $\int_T |L_n(x)(t) - x(t)|^2 dt \longrightarrow 0$ _as_ $n \longrightarrow \infty$.

From this and other similar theorems we derive the convergence in measure of some interpolation procedures.

4.2. THEOREM. _If_ $w(t) \geq w_0 > 0$ _for all_ $t \in T$, _then for each_ x _in_ $C(T)$ _we have_ $L_n(x) \longrightarrow x$ _in measure on_ T _as_ $n \longrightarrow \infty$.

Proof. Let $\sigma > 0$ and $\varepsilon > 0$. Put $E_{\sigma,n} = \{t \in T: |L_n(x)(t) - x(t)| \geq \sigma\}$ and apply Theorem 4.1 to obtain an $n_0 \in N$ such that $\int_T |L_n(x)(t) - x(t)|^2 dt < \varepsilon\sigma^2$ for all $n \geq n_0$. Then $\mu(E_{\sigma,n}) < \varepsilon$ for all $n \geq n_0$ since $\sigma \cdot \mu(E_{\sigma,n}) \leq \int_{E_{\sigma,n}} |L_n(x)(t) - x(t)|^2 dt \leq \int_T |L_n(x)(t) - x(t)|^2 dt < \varepsilon\sigma^2$. ∎

4.3. THEOREM. _If_ $w(t) = (1-t)^\alpha (1+t)^\beta$ _is the Jacobi weight with_ $\alpha = \beta = \frac{1}{2}$ _or_ $\alpha = \beta = \frac{3}{2}$, _and the_ $n+2$ _nodes on the_ n-_th row of_ $\mathfrak{M}$ _are the zeros of the polynomial_ $(1-t^2)\omega_n^{(\alpha,\beta)}(t)$, _then for each_ $x \in C(T)$ _we have_ $L_{n+2}(x) \longrightarrow x$ _in measure on_ T _as_ $n \longrightarrow \infty$.

Proof. A. K. Varma and P. Vértesi [9] proved that, if $p > 0$ and $x \in C(T)$, then $\int_T |L_{n+2}(x)(t) - x(t)|^p (1-t^2)^{-1/2} dt \longrightarrow 0$ as $n \longrightarrow \infty$. Using this for a fixed $p > 0$ and remarking that $(1-t^2)^{-1/2} \geq 1$ for all $t \in T$, the argument in the proof of Theorem 4.2 applies. ∎

4.4. THEOREM. _Suppose that_ $w(t) = (1-t)^\alpha (1+t)^\beta$ _where_ $\alpha = \beta > 0$, _or_ $0 < \alpha \leq \frac{1}{2}$ _and_ $\beta = -\alpha$. _Then for each_ $\varepsilon > 0$, $\varepsilon < 1$, _and each_ x _in_ $C(T)$ _we have_ $L_n(x) \longrightarrow x$ _in measure on_ $[-1+\varepsilon, 1-\varepsilon]$, _and on_ $[-1, 1-\varepsilon]$, _respectively, as_ $n \longrightarrow \infty$.

Proof. R. Askey [1] showed that, if $0 < p < \frac{4(\alpha+1)}{2\alpha+1}$, then

$$\int_T |L_n(x)(t) - x(t)|^p (1-t)^\alpha (1+t)^\beta dt \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty .$$

Denote by w_o a positive number with $w(t) \geq w_o$ for all $t \in [-1+\varepsilon, 1-\varepsilon]$ when $\alpha = \beta > 0$, or for all $t \in [-1, 1-\varepsilon]$ when $0 < \alpha \leq \frac{1}{2}$ and $\beta = -\alpha$, respectively. Then use the cited result of Askey for a fixed p and apply the same argument as in the proof of Theorem 4.2. ∎

Other results concerning the convergence in measure of the interpolation procedures on every compact interval in $\mathbb{R}$ can be derived from the recent mean convergence theorems on the whole real axis obtained in [4].

5. Divergence properties of simple quadrature formulas. Denote by T a compact interval in $\mathbb{R}$, by $(m_n)_{n\in N}$ a strictly increasing sequence of natural numbers, by $\mathfrak{M} = \{(t_n^1,\dots,t_n^{m_n}) : a \leq t_n^1 < \dots < t_n^{m_n} \leq b,\ n \in N\}$ a matrix of nodes, by $\mathcal{C} = \{(c_n^1,\dots,c_n^{m_n}) : c_n^i \in K,\ i \in \{1,\dots,n\},\ n \in N\}$ a matrix of coefficients, and by $v: T \longrightarrow K$ a function with finite variation. Frequently, the function v is given by $v(t) = \int_a^t w(s)ds$, where w is a weight on T, and the coefficients c_n^i are given by the "interpolatory formulas"

(5) $c_n^i = \int_T w(t)\ell_n^i(t)dt$, where ℓ_n^i are defined in Section 4.

By a quadrature formula associated with the above data we mean the equality

(6) $\int_T x(t)dv(t) = Q_n(x) + R_n(x)$, $x \in C(T)$, where $Q_n(x) = \sum_{i=1}^{m_n} c_n^i x(t_n^i)$.

When the coefficients c_n^i are defined by (5), then for all polynomial x we have $R_n(x) = 0$ whenever n is sufficiently large. Therefore, as in Sections 3 and 4 we put the problem: determine the sequences $(m_n)_{n\in N}$, the matrices $\mathfrak{M}$ and $\mathcal{C}$, the functions v and the classes of functions x in $C(T)$ such that

$R_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$. If $T = [-1,1]$, $m_n = 2n+1$, $v(t) = t$ for all $t \in T$, $t_n^i = \frac{i}{n}$ and c_n^i are given by (5) for $i \in \{0, \pm 1, \ldots, \pm n\}$, J. Ouspenski (1925) proved that there is a function $x \in C(T)$ such that $R_n(x) \not\longrightarrow 0$ as $n \longrightarrow \infty$. More precise, H. Brass (1977) showed that $x(t) = |t|$ is such a function. Recall the well-known characterization of convergent quadrature formulas:

5.1. THEOREM (G. Pólya [8]). <u>We have</u> $R_n(x) \longrightarrow 0$ <u>for any</u> x <u>in</u> $C(T)$ <u>as</u> $n \longrightarrow \infty$ <u>if and only if the sequence</u> $\left(\sum_{i=1}^{m_n} |c_n^i|\right)_{n \in N}$ <u>is bounded, and</u> $R_n(x) \longrightarrow 0$ <u>for any polynomial</u> x <u>as</u> $n \longrightarrow \infty$.

Introduced by P. J. Davis (1953), the simple quadrature formulas are of the form (6), with $c_n^i = c_i$ and $t_n^i = t_i$, $i \in \{1, \ldots, m_n\}$, where $(c_i)_{i \in N}$ is a sequence in K, and $(t_i)_{i \in N}$, $t_i \in T$, is a sequence whose terms are mutually distinct.

5.2. THEOREM (see [7]). <u>If</u> $v: T \longrightarrow K$ <u>is a nonconstant continuous function with finite variation, then for the simple quadrature formula</u>

(7) $\int_T x(t)\,dv(t) = Q_n(x) + R_n(x)$, $x \in C(T)$, <u>where</u> $Q_n(x) = \sum_{i=1}^{m_n} c_i x(t_i)$,

<u>there exists an</u> $x_o \in C(T)$ <u>such that</u> $R_n(x_o) \not\longrightarrow 0$ <u>as</u> $n \longrightarrow \infty$.

We say that (7) is <u>unboundedly divergent</u> on a subset Y of $C(T)$ if $\sup\{|Q_n(x)| : n \in N\} = \infty$ for each $x \in Y$.

5.3. THEOREM. <u>The formula</u> (7) <u>is unboundedly divergent at a point in</u> $C(T)$ <u>if and only if</u> $\sum_{i=1}^{\infty} |c_i| = \infty$. <u>Moreover, the last equality implies the unbounded divergence of</u> (7) <u>on a superdense set in</u> $C(T)$.

<u>Proof</u>. Remark that Q_n is a linear and continuous functional on $C(T)$ and that

(8) $$\|Q_n\| = \sum_{i=1}^{m_n} |c_i|.$$

If (7) is unboundedly divergent at a point $x \in C(T)$, then from $|Q_n(x)| \leq \|Q_n\| \cdot \|x\|$ and (8) we derive $\sum_{i=1}^{\infty} |c_i| = \sup\{\|Q_n\| : n \in N\} = \infty$. Conversely, if $\sum_{i=1}^{\infty} |c_i| = \infty$, from Theorem 2.1 and (8) it follows that the set $S = \{x \in C(T): \sup\{|Q_n(x)| : n \in N\} = \infty\}$ is superdense in $C(T)$. Since $S \neq \emptyset$, there is an x at which (7) is unboundedly divergent. ∎

5.4. COROLLARY. Suppose that the hypotheses of Theorem 5.2 are satisfied and that $R_n(x) \longrightarrow 0$ for all polynomial x as $n \longrightarrow \infty$. Then (7) is unboundedly divergent on a superdense set in $C(T)$.

Proof. If $R_n(x) \longrightarrow 0$ for all polynomial x as $n \longrightarrow \infty$, from Theorems 5.1 and 5.2 we derive $\sum_{i=1}^{\infty}|c_i| = \infty$, so that Theorem 5.3 applies. ∎

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