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THE PRINCIPLE OF MAJORANT IN SOLVING OF NONLINEAR OPERATORIAL EQUATIONS WHICH DEPENDS ON ONE PARAMETER, DEFINED IN FRECHET SPACES

by

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1. The aim of this paper is to given some existence conditions for the solution of a nonlinear operatorial equation which depends of one parameter, using the Kantorovici's principle of majorant [1]. According to this, a real equation is associated whith a given operatorial equation, called 'majorant', and based on the hypothesis used for this majorant equation, conclusions for the operatorial equation are established.

2. Let us consider the operator equation

$$P(x,a)=\theta \tag{1}$$

where $P: X\times M \to X$ is a nonlinear continuous operator, X is a Fréchet space [2], M is a quasinormed space [3] and $\theta \in X$ being the null element.

Suppose that the equation (1) is majorized by the equation

$$Q(z,b)=0 \tag{1'}$$

where $Q: D \subset \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $D=[z_0,z'] \times [b_0,b']$, is a real continuous function of two real variables.

In order to approximate the solution of the equation (1) we shall use the iterative method, known under the name 'method analogous to the method of the tangent parabolas',

$$\begin{aligned} x_{n+1}(a) &= x_n(a) - \\ &- \Lambda_{n,a}(I-[x_n,x_{n-1},x_{n-2};P^{(a)}]\Lambda_{n-1,a}P(x_{n-1},a)\Lambda_{n,a})^{-1}P(x_n,a) \\ &\qquad (n=0,1,2,\dots) \end{aligned} \tag{2}$$

where x_{-2},x_{-1},x_0 are given elemnts in X,

$$\Lambda_{i,a} = [x_i,x_{i-1};P^{(a)}]^{-1}$$
$$(i=n,n-1)$$

is the inverse of the first partial divided difference [4] of the operator P in the points (x_n,x_{n-1}), respectively (x_{n-1},x_{n-2}) and $[x_n,x_{n-1},x_{n-2};P^{(a)}]$ is the second order partial divided differen-

…se of the operator P in (x_n, x_{n-1}, x_{n-2}).

To find the solution $z(b)$ of the majorant equation, we consider the iteration

$$z_{n+1}(b) = z_n(b) -$$

(2')

$$\frac{[z_{n-1}, z_{n-2}; Q^{(b)}]\, Q(z_n, b)}{[z_n, z_{n-1}; Q^{(b)}][z_{n-1}, z_{n-2}; Q^{(b)}] - [z_n, z_{n-1}, z_{n-2}; Q^{(b)}]\, Q(z_{n-1})}$$

where $[z', z''; Q^{(b)}]$ and $[z', z'', z'''; Q^{(b)}]$ the first and second partial divided difference of Q are denoted.

In the paper [5] in the case of equation

(1'') $$P(x) = \theta$$

majorized by the real equation

(2'') $$Q(z) = 0$$

the following theorem was proved:

THEOREM A. If equation (1') has the equation (2'') as a majorant and if the following conditions are satisfied for approximations x_0, x_{-1}, x_{-2}, respectively z_0, z_{-1}, z_{-2}:

1. There is $\Lambda = -[x^{(1)}, x^{(2)}; P]^{-1}$

   for every $x^{(i)} \in S$, $i=1,2$, S being defined by the inequality

   $$\|x - x_0\| \leq z' - z_0 \subset I$$, and we have $\|x_i - x_0\| \leq z_0 - z_i$, $i=-1,-2$;

2. $\|\Lambda P(x_i)\| \leq B Q(z_i)$, $i=0,-1,-2$ and

$$B \geq -\frac{1}{[z^{(1)}, z^{(2)}; Q]}$$

3. $\|\Lambda [x^{(1)}, x^{(2)}, x^{(3)}; P]\| \leq B[z^{(1)}, z^{(2)}, z^{(3)}; Q]$

   $\|\Lambda [x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}; P]\| \leq B[z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}; Q]$;
   $x^{(i)} \in S$, $i=\overline{1,4}$

4. $[z^{(1)}, z^{(2)}; Q]\,[z^{(2)}, z^{(3)}; Q] < [z^{(1)}, z^{(2)}, z^{(3)}; Q]\, Q(z^{(2)})$

then the equation (1) has a solution x^*, which is the limit of sequence given by (2), when the bounding

$$\| x^{*} - x_0 \| \le z^{*} - z_0 .$$

In the paper we shall the notation:

$$[x^{(1)},x^{(2)};a^{(1)},a^{(2)};P^{(a;x)}] =$$

$$= [a^{(1)},a^{(2)};[x^{(1)},x^{(2)};P^{(a)}]^{(x)}] ;$$

$$[x^{(1)},x^{(2)},x^{(3)};a^{(1)},a^{(2)};P^{(a^2;x)}] =$$

$$= [a^{(1)},a^{(2)};[x^{(1)},x^{(2)},x^{(3)};P^{(a^2)}]^{(x)}] ;$$

$$[x^{(1)},x^{(2)},x^{(3)};P^{(a^2)}] = [x^{(1)},x^{(2)};[x^{(2)},x^{(3)};P^{(a)}]^{(a)}]$$

for the partial divided diference of the operator partial divided difference.

In the case of equation (1), we prove the following

THEOREM 1. If for the initial approximations $x_0, x_{-1}, x_{-2} \in X$, respectively $z_0, z_{-1}, z_{-2} \in [z_0, z']$ and for the initial values a_0, b_0 for the parameters a,b, the following conditions are satisfied:

1°. The operator $\Lambda_{a_0} = [x^{(1)}, x^{(2)}; P^{(a_0)}]^{-1}$ exists with

$$\|\Lambda_{a_0}\| \le - \frac{1}{[z^{(1)},z^{(2)};Q^{(b_0)}]} \le B_{b_0}, \forall x^{(i)} \in S, i=1,2,$$

$$S=\{x \mid \|x-x_0\| \le z^{*}-z_0\} \text{ and } \|x_s-x_0\| \le z_0-z_s, s=-2,-1;$$

2°. $\|\Lambda P(x_i,a_0)\| \le BQ(z_i,b_0)$, $i=-2,-1,0$ and

$$B > - \frac{1}{[z^{(1)},z^{(2)};Q^{(b)}]} ;$$

3°. $\|\Lambda[x^{(1)},x^{(2)},x^{(3)};P^{(a_0^2)}] \le B[z^{(1)},z^{(2)},z^{(3)};Q^{(b_0^2)}$ and

$$\|\Lambda[x^{(1)},x^{(2)},x^{(3)},x^{(4)};P^{(a_0^3)}]\| \le$$

$$\le B[z^{(1)},z^{(2)},z^{(3)},z^{(4)};Q^{(b_0^3)}]$$

$$\forall x^{(i)} \in S', z^{(i)} \in z_0 - z', i=1,2,3,4;$$

4°. $$\|\Lambda[a^{(1)},a^{(2)};P^{(x_0)}]\| \le B[b^{(1)},b^{(2)};Q^{(z_0)}]$$

$$\forall a^{(i)} \in S_1 = \{a \mid \|a-a_0\| \le b-b_0 \le b'-b_0\}, i=1,2;$$

5°. $\|\Lambda[x(1),x(2)|a(1),a(2);P(a|x)] \le$

$$\le B[z^{(1)},z^{(2)}|b^{(1)},b^{(2)};Q^{(b|z)}] ;$$

$$\|\Lambda[x^{(1)},x^{(2)},x^{(3)}|a^{(1)},a^{(2)};P^{(a^2|x)}]\| \le$$

$$\le B[z^{(1)},z^{(2)},z^{(3)}|b^{(1)},b^{(2)};Q^{(b^2|z)}]$$

$$\forall\ x^{(i)} \in S,\ i=1,2,3,\ a^{(j)} \in S_1,\ j=1,2 ;$$

6°. $\|\Lambda[x^{(1)},x^{(2)},x^{(3)},x^{(4)}|a^{(1)},a^{(2)};P^{(a^3|x)}]\| \le$

$$\le \|(B[z^{(1)},z^{(2)},z^{(3)},z^{(4)}|b^{(1)},b^{(2)};Q^{(b^3|z)}]$$

$$\forall\ x^{(i)} \in S,\ i=\overline{1,4} \text{ and } a^{(j)} \in \sigma,\ j=1,2$$

then from the existence of the solution $z^*(b) \in [z_0,z']$, for any $b \in [b_0,b']$, of equation (1'') which is the limit of the sequence genereted by (2'), it results the unique solution $x^*(a)$ of equation (1), for any $a \in S_1$, solution which is the limit of the sequence generated by the iterativ method (1), the convergence order being given by the inequality

$$(3) \qquad \|x^*(a)-x_0\| \le z^*(b)-z_0 .$$

PROOF. The conditions 1°–3° of Theorem 1 can by applied in the case of equations independent of parameters, i.e. for equation such as $P(x,a_0)=0$, respectively $Q(z,b_0)=0$, a_0 and b_0 being fixed.

For these equations the existence of a solution $x^*(a_0)$ results from the Theorem A.

We prove that these conditions of Theorem A are satisfied for any $a \in S_1$ and $b \in [b_0,b']$.

a). Because

$$I+\Lambda_{a_0}[x^{(1)},x^{(2)};P^{(a)}] =$$

$$= \Lambda_{a_0}(\ [x^{(1)},x^{(2)};P^{(a)}] - [x^{(1)},x^{(2)};P^{(a_0)}]\ ) =$$

$$= \Lambda_{a_0}(\ x^{(1)},x^{(2)}|a,a_0;P^{(a|x)}](a-a_0)),$$

taking into account the condition 5° of Theorem 1, we have ,

$$\|I+\Lambda_{a_0}[x^{(1)},x^{(2)};P^{(a)}]\| \le$$

$$\le B_{a_0}([z^{(1)},z^{(2)}|b^{(1)},b^{(2)};Q^{(b|z)}](z-z_0)) =$$

$$= B_{a_0}(\ [z^{(1)},z^{(2)};Q^{(b)}] - [z^{(1)},z^{(2)};Q^{(b_0)}]\ ) =$$

$$= 1 - \frac{[z(1),z(2);Q(b)]}{[z^{(1)},z^{(2)};Q^{(b_0)}]} = q.$$

By hypotheses we have $[z^{(1)},z^{(2)};Q^{(b_0)}] < 0$ and from the existence of the solution $z(b) \in [z_0,z']$, $\forall b \in [b_0,b']$, it results

$$[z^{(1)},z^{(2)};Q^{(b)}] < 0 .$$

Then, $q<1$ and from Banach's Theorem it follows the existence of the operator

$$H^{-1} = (I-[I+\Lambda_{a_0}[x^{(1)},x^{(2)};P^{(a)}]^{-1}) =$$

$$= -\Lambda_{a_0}[x^{(1)},x^{(2)};P^{(a)}]^{-1}.$$

It results then the existence of

$$H^{-1}\Lambda_{a_0} = -[x^{(1)},x^{(2)};P^{(a)}]^{-1} = \Lambda_a$$

so the condition 1 of Theorem A is verified.

b). To prove that in the conditions of Theorem 1 the condition 2 of Theorem A is verified, we consider

$$\|\Lambda P(x_0,a)\| = \|\Lambda P(x_0,a)-\Lambda P(x_0,a_0)+\Lambda P(x_0,a_0)\| \le$$

$$\le \|\Lambda P(x_0,a_0)\|+\|\Lambda[a,a_0;P^{(x_0)}](a-a_0)$$

which, using the conditions 2^{o} and 4^{o}, may be writen

$$\|\Lambda P(x_0,a)\| \le BQ(z_0,b_0)+B[b,b_0;Q^{(z_0)}](b-b_0) =$$

$$= BQ(z_0,b_0)+B(Q(z_0,b)-Q(z_0,b_0)) =$$

$$= BQ(z_0,b).$$

In the same way we can obtaine

$$\|\Lambda P(x_i,a)\| \le BQ(z_i,b)$$

for $i=-2,-1$.

c). We consider the relation

$$\Lambda[x^{(1)},x^{(2)},x^{(3)};P^{(a^2)}] =$$

$$= \Lambda[x^{(1)},x^{(2)},x^{(3)};P^{(a_0^2)}]+\Lambda[x^{(1)},x^{(2)},x^{(3)};P^{(a^2)}]-$$

$$-\Lambda[x(1),x(2),x(3);P(a_0^2)] =$$

$$=\Lambda[x^{(1)},x^{(2)},x^{(3)};P^{(a_0^2)}] +$$

$$+\Lambda[x^{(1)},x^{(2)},x^{(3)}|a,a_0;P^{(a^2|x)}[(a-a_0)$$

and by the conditions $4^0, 5^0$, we have

$$\|\Lambda[x^{(1)},x^{(2)},x^{(3)};P^{(a^2)}]\| \le$$

$$\|B[z^{(1)},z^{(2)},z^{(3)};Q^{(b_0^2)}]+\|B[z^{(1)},z^{(2)},z^{(3)}|b,b_0;Q^{(b^2|z)}]_{(b-b_0)}\| \le$$

$$\le B[z^{(1)},z^{(2)},z^{(3)};Q^{(b_0^2)}] +$$

$$+ B[z^{(1)},z^{(2)},z^{(3)};Q^{(b^2)}] - B[z^{(1)},z^{(2)},z^{(3)};Q^{(b_0^2)}] =$$

$$= B[z^{(1)},z^{(2)},z^{(3)};Q^{(b^2)}]$$

hence the first inequality of the condition 3 of Theorem A is verified.

To prove the second inequality, we considering the relation

$$\Lambda[x^{(1)},x^{(2)},x^{(3)},x^{(4)};P^{(a^3)}]=\Lambda[x^{(1)},x^{(2)},x^{(3)},x^{(4)};P^{(a_0^3)}]+$$

$$+\Lambda[x^{(1)},x^{(2)},x^{(3)},x^{(4)};P^{(a^3)}]-\Lambda[x^{(1)},x^{(2)},x^{(3)},x^{(4)};P^{(a_0^3)}]=$$

$$=\Lambda[x^{(1)},x^{(2)},x^{(3)},x^{(4)};P^{(a_0^3)}] +$$

$$+ \Lambda([x^{(1)},x^{(2)},x^{(3)},x^{(4)}|a,a_0;P^{(a^3|x)}](a-a_0)).$$

By the condition 3^0 and 6^0, it follows

$$\|\Lambda[x^{(1)},x^{(2)},x^{(3)},x^{(4)};P^{(a^3)}]\| \le B[z^{(1)},z^{(2)},z^{(3)},z^{(4)};Q^{(b_0^3)}] +$$

$$+ B([z^{(1)},z^{(2)},z^{(3)},z^{(4)}|b,b_0;Q^{(b^3|z)}](b-b_0)) =$$

$$= B[z^{(1)},z^{(2)},z^{(3)},z^{(4)};Q^{(b_0^3)}] + B[z^{(1)},z^{(2)},z^{(3)},z^{(4)};Q^{(b^3)}] -$$

$$- B[z^{(1)},z^{(2)},z^{(3)},z^{(4)};Q^{(b_0^3)}] = B[z^{(1)},z^{(2)},z^{(3)},z^{(4)};Q^{(b^3)}].$$

The hypothesis of Theorem A being true, it resulte the conclusions of Theorem 1.

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