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SOME SPLINE APPROXIMATION BASED ON GENERALIZED GAUSS-TYPE QUADRATURE FORMULAS

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Summary. One considers the problem of approximating a given function f on $[0,1]$ by a spline function of fixed degree with variable knots. The deficiencies of this spline function are greater or equal than one. The spline approximation is constructed in such a way that it interpolates the value of the function f and some of its derivatives at the end right point of the interval $[0,1]$, and it preserves some moments of the function. The particular case when the deficiency is one in all knots was considered in [3]. As in particular case, the solution of our problem will be obtained by some generalized Gauss quadrature formulas. The elements of the spline approximation are given by the coefficients and the nodes of these quadratures. When the spline approximation exists the error term in spline approximation formula is expressed by the error term from the corresponding quadratures.

Let $\mathcal{S}_{m,r}$ be the space of polynomial spline functions of degree m with the variable knots t_ν, $\nu=\overline{1,n}$, $0<t_1<t_2<\ldots<t_n<1$ and the deficiency given by the positive vector $r=(r_1,r_2,\ldots,r_n)$. One considers the problem of approximating a function $f\in C^{m+1}[0,1]$ by a spline function $s_f\in\mathcal{S}_{m,r}$ so that it verifies the interpolation conditions

$$(1)\qquad s_f^{(j)}(1)=f^{(j)}(1),\quad j=\overline{0,m}$$

and the moment functional conditions

$$(2)\qquad \int_0^1 t^j s_f(t)\,dt=\int_0^1 t^j f(t)\,dt\ ,\ j=\overline{0,N+n-1},$$

where $N=r_1+r_2+\ldots+r_n$, i.e. the sum of deficiencies or equivalently the sum of multiplicities of the knots t_ν, $\nu=\overline{1,n}$.

A spline function $s\in\mathcal{S}_{m,r}$ can be written in the form

$$(3)\qquad s(t)=p_m(t)+\sum_{\nu=1}^{n}\sum_{\mu=0}^{r_\nu-1} a_{\nu\mu}(t_\nu-t)_+^{m-\mu}$$

and also

$$(4)\qquad s(t) = p_m(t) + \sum_{\nu=1}^{n}\sum_{\mu=0}^{r_\nu-1}\alpha_{\nu\mu}\left[\frac{d^\mu}{dx^\mu}(x-t)_+^m\right]_{x=t_\nu},$$

where $p_m(t)$ is a polynomial of degree m, we denote it $p_m \in \mathcal{P}_m$, and $a_{\nu\mu} = m(m-1)\dots(m-\mu+1)\alpha_{\nu\mu} = m^{[\mu]}\alpha_{\nu\mu}$.

Using one of the two representations for s_f, we observe that the interpolation conditions (1) are reduced to

$$(5)\qquad p_m^{(j)}(1) = f^{(j)}(1),\quad j=\overline{0,m},$$

which determine uniquely the polynomial part $p_m(t)$. Accordingly, our problem is reduced to the determination of the coefficients $\alpha_{\nu\mu}$, $\nu=\overline{1,n}$, $\mu=\overline{0,r_\nu-1}$, and the knots t_ν, $\nu=\overline{1,n}$, from the conditions

$$(6)\ \sum_{\nu=1}^{n}\sum_{\mu=0}^{r_\nu-1} a_{\nu\mu}\int_0^1 t^j(t_\nu-t)_+^{m-\mu}\,dt = \int_0^1 t^j\left[f(t)-p_m(t)\right]dt,\ j=\overline{0,N+n-1}.$$

We have that

$$\int_0^1 t^j(t_\nu-t)_+^{m-\mu}\,dt = \frac{j!\,m!}{(m+j+1)!}\cdot\frac{1}{m^{[\mu]}}\left[\frac{d^\mu}{dt^\mu}(t^{m+1+j})\right]_{t=t_\nu}$$

and on the other hand taking into account (5) and the generalized formula of integration by parts it results successively

$$\int_0^1 t^j\left[f(t)-p_m(t)\right]dt = \frac{j!}{(m+j+1)!}\Bigg\{\sum_{i=0}^{m}(-1)^i\left[f^{(i)}(1)-p_m^{(i)}(1)\right]\cdot$$

$$\cdot\left[\frac{d^{m-i}}{dt^{m-i}}(t^{m+1+j})\right]_{t=1} + (-1)^{m+1}\int_0^1\left[f^{(m+1)}(t)-p_m^{(m+1)}(t)\right]t^{m+1+j}dt\Bigg\} =$$

$$= \frac{j!\,m!}{(m+j+1)!}\cdot\frac{(-1)^{m+1}}{m!}\int_0^1 f^{(m+1)}(t)\,t^{m+1+j}dt.$$

Based on the last two relations we obtain that (6) is equivalent to

$$(7)\ \sum_{\nu=1}^{n}\sum_{\mu=0}^{r_\nu-1}\alpha_{\nu\mu}\left[\frac{d^\mu}{dt^\mu}(t^{m+1+j})\right]_{t=t_\nu} = \frac{(-1)^{m+1}}{m!}\int_0^1 f^{(m+1)}(t)\,t^{m+1+j}dt,$$

$$j=\overline{0,N+n-1}.$$

One defines the linear functionals $\mathcal{L}_0$ and $\mathcal{L}$ by

$$(8)\qquad \mathcal{L}_0(g) = \sum_{\nu=1}^{n}\sum_{\nu=0}^{r_\nu-1}\alpha_{\nu\mu}\,g^{(\mu)}(t_\nu)$$

and

(9) $$\mathcal{L}(g) = \int_0^1 g(t)d\lambda(t),$$

where $d\lambda(t) = \frac{(-1)^{m+1}}{m!} f^{(m+1)}(t)dt$. So that (7) can be written in the equivalent form

(10) $$\mathcal{L}_o(t^{m+1}.p(t)) = \mathcal{L}(t^{m+1}.p(t)), \quad p \in \mathcal{P}_{N+n-1}.$$

Let consider the inner product defined by means of the functional $\mathcal{L}$

(11) $$(u,v) = \mathcal{L}(t^{m+1}.u(t)v(t)) = \int_0^1 u(t)v(t)d\lambda(t).$$

One considers (if it exists) the monic polynomial $\omega_N = \omega_N(.;\mathcal{L})$ of degree N orthogonal with respect to inner product (11) to all polynomials of lower degree than n.

THEOREM 1. <u>If</u> f $C^{m+1}[0,1]$ <u>then there exists a unique spline function</u> $s_f \in \mathcal{S}_{m,r}$, <u>which satisfies (1) and (2) if and only if the orthogonal polynomial</u> $\omega_N = \omega_N(.;\mathcal{L})$ <u>uniquely exists and it has n real zeros</u> τ_ν <u>with the multiplicities</u> r_ν, $\nu = \overline{1,n}$, <u>and</u> $0 < \tau_1 < \tau_2 < \ldots < \tau_n < 1$. <u>Moreover in this case we have that</u> $t_\nu = \tau_\nu$, $\nu = \overline{1,n}$, <u>and the coefficients</u> $\alpha_{\nu\mu}$, $\nu = \overline{1,n}$, $\mu = \overline{0,r_\nu - 1}$, <u>can be determinated from the conditions</u> $\mathcal{L}_o(t^{m+1+j}) = \mathcal{L}(t^{m+1+j})$, $j = \overline{0,N-1}$.

<u>Proof</u>. When the spline approximation s_f exists uniquely we can define the linear functional $\mathcal{L}_o$ by (8). Then one considers the monic polynomial $\omega_N(t) = (t-t_1)^{r_1}(t-t_2)^{r_2}\ldots(t-t_n)^{r_n}$. Because (10) is satisfied we have successively for any polynomial $p \in \mathcal{P}_{n-1}$

$$(\omega_N,p) = \mathcal{L}(t^{m+1}.\omega_N(t)p(t)) = \mathcal{L}_o(t^{m+1}.\omega_N(t)p(t)) = 0 \;,$$

i.e. ω_N is orthogonal to all polynomials of lower degree than n.

For the sufficiency one knows that any polynomial $p \in \mathcal{P}_{N+n-1}$ can be written in the form $p(t) = \omega_N(t)q(t) + r(t)$. Here ω_N is the monic polynomial of degree N orthogonal with respect to inner product (11) to all polynomials of degree less than n, and q and r are two polynomials of degree n-1 and N-1 respectively. Accordingly with this we have

$$\mathcal{L}(t^{m+1}.p(t)) = \mathcal{L}(t^{m+1}.\omega_N(t)q(t)) + \mathcal{L}(t^{m+1}.r(t)) = \mathcal{L}(t^{m+1}.r(t)).$$

On the other hand when the linear $\mathcal{L}_o$ has the definition formula (10) we have

$$\mathcal{L}_o(t^{m+1}.p(t)) = \mathcal{L}_o(t^{m+1}.\omega_N(t)q(t)) + \mathcal{L}_o(t^{m+1}r(t)) = \mathcal{L}_o(t^{m+1}.r(t)).$$

So that the condition (10) will be satisfied when $\mathcal{L}_0(t^{m+1+j}) = \mathcal{L}(t^{m+1+j})$, $j=\overline{0,N-1}$. This linear system with the unknowns $\alpha_{\nu\mu}$, $\nu=\overline{1,n}$, $\mu=\overline{0,r_\nu-1}$, has a generalized Vandermonde. Therefore, this system has a unique solution. This completes the proof of the theorem.

This theorem relates the solution of our problem to the construction of some generalized Gauss-Radau quadrature formula and its corresponding generalized Gauss-Christoffel quadrature.

Let be the generalized Gauss-Radau quadrature formula

$$(12)\qquad \int_0^1 g(t)\,d\lambda(t) = \sum_{i=0}^{m} A_i g^{(i)}(0) + \sum_{\nu=1}^{n}\sum_{\mu=0}^{r_\nu-1} \lambda_{\nu\mu} g^{(\mu)}(\tau_\nu) + R(g;d\lambda),$$

where $d\lambda(t)$ is the above specified measure and τ_ν, $\nu=\overline{1,n}$, are the zeros of the monic polynomial ω_N (if it exists and it has the real zeros $0<\tau_1<\tau_2<\ldots<\tau_n<1$ with the multipilicities r_ν, $\nu=\overline{1,n}$, respectively). The degree of exactness of this quadrature formula is N+n+m, i.e. $R(g;d\lambda)=0$, when $g\in\mathcal{P}_{N+n+m}$. A generalized Gauss-Christoffel quadrature formula corresponds to (12), namely

$$(13)\qquad \int_0^1 g(t)\,d\sigma(t) = \sum_{\nu=1}^{n}\sum_{\mu=0}^{r_\nu-1} \sigma_{\nu\mu} g^{(\mu)}(\tau_\nu) + R(g;d\sigma),$$

where $d\sigma(t)=t^{m+1}d\lambda(t)$, the nodes τ_ν, $\nu=\overline{1,n}$, are the same as in (12), and

$$\sigma_{\nu\mu} = \sum_{k=0}^{r_\nu-\mu+1} \lambda_{\nu,\mu+k} \binom{\mu+k}{k} (m+1)^{[k]} \tau_\nu^{m+1-k}.$$

The degree of exactness of (13) is N+n-1.

In (12) one takes $g(t)=t^{m+1}.p(t)$, $p\in\mathcal{P}_{N+n-1}$, then $R(t^{m+1}.p(t);d\lambda(t))=0$, or equivalently

$$(14)\qquad \sum_{\nu=1}^{n}\sum_{\mu=0}^{r_\nu-1} \lambda_{\nu\mu}\left[\frac{d^\mu}{dt^\mu}(t^{m+1+j})\right]_{t=\tau_\nu} = \int_0^1 t^{m+1+j}\,d\lambda(t),\quad j=\overline{0,N+n-1}.$$

The relations (14) are identical with (7) when $\alpha_{\nu\mu}=\lambda_{\nu\mu}$, $\nu=\overline{1,n}$, $\mu=\overline{0,r_\nu-1}$, and $t_\nu=\tau_\nu$, $\nu=\overline{1,n}$. Taking into account these remarks we have the following theorem.

THEOREM 2. <u>If the conditions of Theorem 1 are satisfied then the elements of the spline approximating function</u> s_f <u>are given by</u> $t_\nu=\tau_\nu$, $\alpha_{\nu\mu}=\lambda_{\nu\mu}$, $\nu=\overline{1,n}$, $\mu=\overline{0,r_\nu-1}$.

When the function f is completely monotonic on [0,1] and r_ν, $\nu=\overline{1,n}$, are odd positive integers then the measures $d\lambda(t)$ and

$d\sigma(t)$ are positive. In this case the existence and the uniqueness of the two quadrature (13) and (14) are assured [4,5,6,7]. Consequently, the spline approximation s_f exists uniquely and a spline approximation formula is obtained.

THEOREM 3. In the conditions of Theorem 1 we have for any $x \in (0,1)$

$$(15) \qquad e_f(x) = f(x)-s_f(x) = R((t-x)_+^m; d\lambda(t)) = R(\frac{(t-x)_+^m}{t^m}; d\sigma(t)).$$

Proof. By Taylor's formula we have

$$(16) \qquad f(x) = \sum_{i=0}^{m} \frac{f^{(i)}(1)}{i!}(x-1)^i + \int_0^1 (t-x)_+^m d\lambda(t).$$

On the other hand from the representation (4) and taking into account the relations (5) we have from Theorem 2

$$(17) \qquad s_f(x) = \sum_{i=0}^{m} \frac{f^{(i)}(1)}{i!}(x-1)^i + \sum_{\nu=1}^{n} \sum_{\mu=0}^{r_\nu-1} \lambda_{\nu\mu}\left[\frac{d^\mu}{dt^\mu}(t-x)_+^m\right]_{t=\tau_\nu}.$$

Using (16) and (17) it results that

$$e_f(x) = f(x)-s_f(x) = \int_0^1 (t-x)_+^m d\lambda(t) - \sum_{\nu=1}^{n} \sum_{\mu=0}^{r_\nu-1} \lambda_{\nu\mu}\left[\frac{d^\mu}{dt^\mu}(t-x)_+^m\right]_{t=\tau_\nu} =$$

$$= R((t-x)_+^m; d\lambda(t)),$$

i.e. the first part of (15).

For the second part we observe that

$$R(g;d\sigma) = R(g(t)t^{m+1}; d\lambda(t)).$$

Consequently, one can write

$$e_f(x) = R((t-x)_+^m; d\lambda(t)) = R(\frac{(t-x)_+^m}{t^{m+1}}; d\sigma(t))$$

which completes the proof.

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