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Q - OPTIMAL FORMULAS OF QUADRATURE WITH WEIGHT FUNCTION OF THE LAGUERRE TYPE

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In this paper the Q-optimal formulas of quadrature (see [5], [1]) are constructed, refferring to the weight function

$w:(0,+\infty)\to(0,+\infty)$, $w(x)=x^{\alpha}e^{-x}$, $\alpha>-1$, named the weight function of the Laguerre type.

We consider the quadrature formulas of the type

$$(1)\quad \int_0^{\infty} x^{\alpha}e^{-x}f(x)dx=\sum_{i=1}^{n} A_i f(x_i)+R_{n-1}(f),\ \alpha>-1,$$

so that the algebrical degree of exactity to be n-1.

We have

$$A_i=\int_0^{\infty} x^{\alpha}e^{-x}\,l_i(x)dx,\quad i=\overline{1,n},$$

$$(2)\quad R_{n-1}^{(f)}=\int_0^{\infty} x^{\alpha}e^{-x}u(x)[x_1,x_2,\dots,x_n,x;f]\,dx,$$

where

$$l_i(x)=\frac{u(x)}{u'(x_i)(x-x_i)},\quad i=\overline{1,n},$$

$$u(x)=\prod_{i=1}^{n}(x-x_i)$$

and $[x_1,\dots,x_n,x;f]$ is the divided difference of f(x) on the nodes $x_1,x_2,\dots,x_n,x$ (see [4]).

Consider the function Q(x) defined by the relation

$$(3)\quad Q(x)=\int_0^{x} x^{\alpha}e^{-x}u(x)dx$$

or, equivalently,

$$(4)\quad Q'(x)=x^{\alpha}e^{-x}u(x)$$

with $Q(0)=0$, $Q(+\infty)=0$ (see [3],[5],[1]).

From (4) and from (2) we obtain

$$R_{n-1}(f)=\int_0^{\infty} Q'(x)\left[x_1,\dots,x_n,x;f\right]dx.$$

Here, using integration by parts, we find

$$R_{n-1}(f)=-\int_0^{\infty}\left[x_1,x_2,\dots,x_n,x,x;f\right]Q(x)dx.$$

If $Q(x)$ is of constant sign in $(0,+\infty)$, the theorem of mean for integrals is applicable, and there follows

$$R_{n-1}(f)=-\frac{f^{(n+1)}(\xi)}{(n+1)!}\int_0^{+\infty}Q(x)dx,\quad \xi\in(0,+\infty).$$

Definition. We define the Q - optimal quadrature formula with weight of Laguerre type, a quadrature formula of the type (1) for which

$$(5)\quad \left|\int_0^{\infty}Q(x)dx\right|=\min \quad \text{(see [5],[1])}.$$

Such formulas are useful in the study of semidefinition of the Peano kernel for several quadrature formulas (see [5],[1]).

For α natural number, from (3) and (4) it results

$$(6)\quad Q(x)=e^{-x}x^{\alpha}P_{\alpha+n-1}(x),$$

where $P_{\alpha+n-1}(x)$ is an arbitrary polynomial of the degree $\alpha+n-1$.

From (6) and (4) it follows

$$(7)\quad u(x)=x^{-\alpha}e^{x}\left[e^{-x}x^{\alpha}P_{\alpha+n-1}(x)\right]'.$$

The function $u(x)$ given by (6) is a polynomial only if $\alpha=0$.

Thus, we are have the theorem:

Theorem 1. For the weight $w(x)=e^{-x}x^{\alpha}$, α natural number, $\alpha\neq 0$, the Q - optimal quadrature formulas don't exist.

In the case $\alpha=0$, from (3) and (6), have

$$\left|\int_0^{\infty}e^{-x}xQ_{n-1}(x)dx\right|=\min.$$

Now, using the lemmas given in [2] the following theorems are demonstrated.

Theorem 2. If $n \equiv 1 \pmod 2$ and $\alpha = 0$, then Q - optimal quadrature formula with weight of Laguerre type has the polynomial of the nodes given by

$$u(x)= Ce^{x}\frac{d}{dx}\left[e^{-x}x(P^{(1)}_{\frac{n-1}{2}}(x)\right]^{2},$$

where $P^{(1)}_{\frac{n-1}{2}}(x)$ is the Laguerre polynomial and C is an arbitrary constant.

The optimal constant of the remainder is given by

$$\int_0^{\infty} Q(x)dx = (\tfrac{n-1}{2})!\,\Gamma(\tfrac{n+3}{2}).$$

Theorem 3. If $n \equiv 0 \pmod 2$ and $\alpha = 0$, then Q - optimal quadrature formula with weight of Laguerre type has the polynomial of the nodes given by

$$u(x)= C\,e^{x}\frac{d}{dx}\left[e^{-x}x^{2}P^{(2)}_{\frac{n-2}{2}}(x)\right]^{2},$$

where $P^{(2)}_{\frac{n-2}{2}}(x)$ is the Laguerre polynomial and C is an arbitrary constant.

The optimal constant of the remainder is given by

$$\int_0^{\infty} Q(x)dx = (\tfrac{n-2}{2})!\,\Gamma(\tfrac{n+4}{2}).$$

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