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A GENERALIZATION OF THE RIEMANN INTEGRAL

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1. The Newton integral

In our previous paper [1] we have defined a notion of almost everywhere primitive for a real function on an interval I of the real axis.

Definition 1.

Let $f:I\to\mathbb{R}$ be given, and $D\subset I$ a Lebesgue null set. A function

$$F : I \setminus D \to \mathbb{R}$$

is called **a.e.primitive** of f on I provided

(i) F is approximately differentiable on $I \setminus D$;

(ii) $F'_{ap} = f$, a.e. on I;

(iii) For each $d \in D \setminus \mathring{I}$ ($\mathring{I}$ denotes the interior of I) we have:

$$\lim_{\substack{\varepsilon\to 0 \\ d+\varepsilon,\, d-\varepsilon\in I\setminus D}}[F(d+\varepsilon)-F(d-\varepsilon)]=0 \qquad (1)$$

If $d \in \mathrm{Fr}\, I$, then the existence of finite $F(d+)$ or $F(d-)$ is assumed.

We say also that f is **a.e. primitivable** on I.

For the purpose of the present paper, we need the Corollary of Theorem 2 from [1]:

THEOREM 1.

Let $F,G : I\setminus D \to \mathbb{R}$ be two a.e. primitives of a given function $f : I \to \mathbb{R}$.

Then

$$F - G = c(const) \text{ on } I \setminus D.$$

Remarks.

1) The a.e. primitive is generally discontinous, but in view of theorem 1 we are able to define a descriptive nonabsolute integral, which includes at least the Riemann, Lebesgue and generalized Riemann integrals;

2) If $F : I\setminus D_1 \to \mathbb{R}$ and $G : I\setminus D_2 \to \mathbb{R}$ are two primitives of a given function $f: I \to \mathbb{R}$ and $D_1 \neq D_2$, we may take only the restricions $F|_{I\setminus D}, G|_{I\setminus D}$, in our considerations, where $D=D_1\cup D_2$.

Thus, without loss of generality, we fix the set D throughout this paper.

Definition 2.

A function $f: [a,b] \to \mathbb{R}$ is said to be **Newton-integrable** on $[a,b]$ if f is a.e. primitivable on $[a,b]$. The Newton-integral, denoted by

$$(N)-\int_a^b f(x)\,dx,$$

is defined by

$$(N)-\int_a^b f(x)\,dx=\tilde{F}(b)-\tilde{F}(a)$$

where

$$\tilde{F}(a)=\begin{cases} F(a) &, \text{if } a\notin D \\ F(a+) &, \text{if } a\in D \end{cases}$$

and

$$\tilde{F}(b)=\begin{cases} F(b) &, \text{if } b\notin D, \\ F(b-) &, \text{if } b\in D, \end{cases}$$

Example 1.

Let $f: [-1,1]\to\mathbb{R}$, $f(0) = 0$,

$$f(x)=\frac{1-x}{x^3}\sin\frac{1}{x}-\cos\frac{1}{x}, \text{for } x>0.$$

and

$$f(x)=\frac{1}{x^2}\sin\frac{1}{x}, for\ x<0.$$

The function F: $[-1,1]\setminus\{0\}\to\mathbb{R}$,

$$F(x)=\cos\frac{1}{x}, if\ x<0$$

and

$$F(x)=(1-x)\cos\frac{1}{x}, if\ x>0,$$

is an a.e. primitive of f on $\mathbb{R}$, since, in this case, D={0}, and

$$\lim_{\varepsilon\to 0}(F(\varepsilon)-F(-\varepsilon))=\pm\lim_{\varepsilon\to 0}\varepsilon\cos\frac{1}{\varepsilon}=0.$$

Hence

$$(N)-\int_{-1}^{1}f(x)\,dx=F(1)-F(-1)=-\cos 1.$$

It is obvious that f does not posses a continous primitive and that f is not Riemann integrable.

The following theorem emphashizes the relationship between the Riemann integral and the Newton integral.

THEOREM 2.

Any Riemann integrable function $f:[a,b]\to\mathbb{R}$ is Newton integrable and

$$(N)-\int_a^b f(x)\,dx=\int_a^b f(x)\,dx.$$

Proof.

Applying the Lebesgue **theorem** [10], we deduce that f is bounded and a.e. continuous on [a,b]. Then the function $F:[a,b]\to\mathbb{R}$, given by

$$F(x)=\int_a^x f(t)\,dt$$

is continous and a.e. differentiable on [a,b] [10] [8].

Let

D= {d∈ [a,b]/F is not differentiable at d}.

Obviously, D contains only all the discontinuity points of f, hence D is a Lebesgue null set.

Since F is continuous, (1) holds, for all $d \in D \cap (a,b)$.

Therefore f is a.e. primitivable, i.e. f is Newton integrable on [a,b].

To prove the second statement, we may assume, any loss of generality, $a,b \notin D$.

Then, by (2), we have

$$(N)-\int_a^b f(x)\,dx = F(b)-F(a) = \int_a^b f(x)\,dx,$$

because F(a) = 0.

Example 2.

Let $f:[0,1]\to\mathbb{R}$ be the characteristic function of the Cantor ternary set $\mathbf{C}_0$, i.e. f(x)=1, when $x \in \mathbf{C}_0$ and f(x)=0 when $x \in [0,1] \setminus \mathbf{C}_0$. Then $F: [0,1]\setminus\mathbf{C}\to\mathbb{R}$, F(x) = c(constant) is an a.e. primitive of f on [0,1].

Hence f is Newton integrable on [0,1] and

$$(N)-\int_0^1 f(x)\,dx = 0.$$

In fact, f=0, a.e on [0,1].

A more general result holds

CORROLARY 1

Let $f,g:[a,b]\to\mathbb{R}$ be two function such that f is Newton integrable and f=g a.e. on [a,b].

Then g is also Newton integrable on [a,b] and

$$(N)-\int_a^b f(x)\,dx = (N)-\int_a^b g(x)\,dx.$$

From theorem 2 we obtain two interesting Leibniz-Newton's formulas.

CORROLARY 2.

Let $f:[a,b]\to\mathbb{R}$ be a strict primitivable function. Then f is Newton integrable on [a,b] and

$$(N)-\int_a^b f(x)\,dx = F(b)-F(a),$$

where F is a strict primitive of f.
(Recall that a strict primitive [1], [2] is a primitive in the usual sense).

CORROLARY 3.

Let $f: [a,b]\to\mathbb{R}$ be a Riemann integrable function. Then f is a.e. primitivable on [a,b] and

$$\int_a^b f(x)\,dx=F(b)-F(a),$$

where F is some a.e. primitive of f.

Remarks.

1) The corrolaries 2 and 3 are generalizations of the Leibniz-Newton's formula for the Riemann integral;

2) A similar result to that of corrolary 3 is given in [8] for the Riemann generalized integral.

It is well-known that, generally, a Riemann integrable function have not a strict primitive.

Example 3.

The function $f: [0,1]\to\mathbb{R}$,

$$f(x)=\frac{1}{x}\cos\frac{1}{x^2}$$

when $x\neq 0$ and $f(0)=0$ is not Lebegue integrable but f is Henstock-Kurzweil (generalized Riemann) integrable [9].

From corrolary 2 we obtain that f is Newton integrable and

$$(N)-\int_0^1 f(x)\,dx=6C\sin\frac{1}{C^2}-3\sin 1,$$

where $C\in(0,1)$ is the intermediary point in the mean value theorem applied to a strict primitive of the continuous function

$$g:[0,1]\to\mathbb{R},\ g(x)=x\sin\frac{1}{x^2},\ \textit{if}\ \neq 0,\ \textit{and}\ g(0)=0.$$

Thus the following question arise: wether any Henstock-Kurzweil integrable function is Newton-integrable.

Using the equivalence betweem the Henstock-Kurzweill integrability and the classical Denjoy-Perron integrability [4], [12], the answer to the above question is given by

THEOREM 3.

Let f: [a,b]→ℝ be a Henstock-Kurzweil integrable function.

Then f is Newton integrable on [a,b] and, moreover, the two integrals are equal.

Proof.

We recall that, if f is Denjoy-Perron integrable, then (see [7]) there a exist a cvasigeneralized absolutely continuous function

$$F:\ [a,b]\to\mathbb{R},$$

such that

$$F' = f \text{ a.e. on } [a,b].$$

Moreover, F(b) − F(a) is the Denjoy-Perron integral of f on [a,b] ([7]).

Since F is a generalized absolutely continuous function, F is continuous on [a,b]. We denote by D the set of all $d\in[a,b]$ such that F is not differentiable at d.

Obviously, D is of Lebesgue measure zero and (1) holds, for each $d\in D\cap(a,b)$.

Hence f is Newton integrable on [a,b] and

$$(N)-\int_a^b f(x)\,dx = F(b)-F(a)$$

which completes the proof ($a,b\notin D$ is assumed).

Remark.

The Newton integral is an effective extension of the Henstock-Kurzweil integral, as shown by

Example 4.

Let f: [−1,1]→ℝ be the function defined by

$$f(x)=\frac{1}{x},$$

when $x\neq 0$ and f(0) = 0.

Then f is not Henstock-Kurzweill integrable on [−1,1], since f is not Henstock-Kurzweil integrable on [−1,0] and [0,1], see [8], exercise 20, p.69.

However, $F:\ [-1,1]\setminus\{0\}\to\mathbb{R}$,

$$F(x)=\ln|x|,$$

is an a.e. primitive of f on [−1,1], since

$$\lim_{\varepsilon\to 0}[F(\varepsilon)-F(-\varepsilon)]=0.$$

Hence f is Newton integrable on [-1,1] and

$$(N)-\int_{-1}^{1} f(x)\,dx=0.$$

2. Basic properties of the Newton integral

The topics treated in this section are all familiar from discussions of the Riemann integral: linearity of the integral as a function of integrands, etc.

The proofs demand mainly familiar arguments of calculus. It does not seem neccessary to give them here.

THEOREM 4.

Let $f: [a,b]\to\mathbb{R}$ be Newton integrable. Let c be a real constant. Then c f and f+g are Newton integrable.
Also

$$(N)-\int_a^b cf(x)\,dx=C\quad (N)-\int_a^b f(x)\,dx$$

and

$$(N)-\int_a^b (f(x)+g(x))\,dx=(N)-\int_a^b f(x)\,dx+(N)-\int_a^b g(x)\,dx$$

This extends to all finite linear combination

$$\sum_{k=1}^{n} c_k f_k$$

as well.

THEOREM 5.

Suppose f is Newton integrable on [a,c] and on [c,b]. Then f is Newton integrable on [a,b] and

$$(N)-\int_a^b f(x)\,dx=(N)-\int_a^c f(x)\,dx-(N)\int_c^b f(x)\,dx$$

Remarks.

1) Recall a well known property of the Riemann, Lebesgue and Henstock-Kurzweil integrals [8]: a function f which is integrable on an interval I is also integrable on each closed subinterval J of I.

The function f defined in example 4 is Newton integrable on [-1,1], but f is not Newton integrable on [0,1] and [-1,0], since

$$\lim_{\varepsilon\to 0} \ln|\varepsilon|=-\infty$$

2) The concept of Newton integrability is important because it is equivalent to a concept of primitivability, i.e. the a.e. primitivability.

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