Buletinul Ştiinţific al Universităţii Baia Mare
Seria B, Matematică-Informatică, vol.VIII(1992),19 - 26

SOME MAXIMUM PRINCIPLES FOR A CLASS OF SINGULAR OPERATORS OF MIXED PARABOLIC-HYPERBOLIC TYPE

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CÎTEVA PRINCIPII DE MAXIM PENTRU O CLASĂ DE OPERATORI DE TIP MIXT PARABOLIC-HIPERBOLIC

Rezumat.

În această lucrare se dau cîteva principii de maxim pentru o clasă de operatori singulari de tip mixt parabolic-hyperbolic, exprimînd condiţiile numai cu ajutorul coeficienţilor.

The purpose of this paper is to give some maximum principles for a class of singular parabolic-hyperbolic type equations of the form

$$(1)\quad Lu:=\begin{cases} L_1(u) \text{ in } D_1 \\ \\ L_2(u) \text{ in } D_2 \end{cases} = 0$$

Here L_1 is the parabolic operator given by

$$(2)\quad L_1(u):= u_{xx} + a_1(x,y)u_x + b_1(x,y)u_y + c_1(x,y)u,$$

where a_1, b_1, $c_1 \in \mathbf{C}(\bar{D}_1)$, and the domain D_1 is defined by

$$(3)\quad D_1 = \{(x,y) \in \mathbb{R}^2 \ / \ -a<x<0,\ 0<y<b,\ a>0,\ b>0\},$$

i.e. D_1 is a rectangle bounded by the perpendicular segments AB, BB_o, B_oA_o, A_oA, where $A_o(-a,b)$, $B_o(0,b)$, $B(0,0,)$, $A(-a,0)$.

The operator L_2 is singular hyperbolic given by

$$L_2(u):=u_{xx}-h^2(x)\,u_{yy}+q\frac{h'(x)}{h(x)}u_x+ph'(x)\,u_y+c_2(x,y)\,u$$

in the domain D_2 bounded by the segment BB_o of Oy- axis and by two characteristics Γ_1, Γ_2 of the operator L_2 through B and B_0, respectively, which is with the positive x-coordonate and one intersect in the point C(a,c).

We denote $D = D_1 \cup D_2 \cup BB_0$, and suppose that the coefficients of the operators L_1 and L_2 fulfill the conditions:

(5) a_1, b_1, c_1 satisfie the Hölder condition in respect to x, again b_1 and in respect to y;

(6) $b_1(x,y) \le -b_0 < 0$, and $c_1(x,y) \le 0$ for any $(x,y) \in \bar{D}_1$;

(7) $h \in \mathbf{C}^2(0,a] \cap \mathbf{C}^1[0,a]$, or (8) $h \in \mathbf{C}^1[0,a]$;

(9) $h(0) = 0$ and $h(x)\, h'(x) > 0$ for $x \in (0,a]$;

(10) $c_2 \in \mathbf{C}(\bar{D}_2 \setminus BB_0)$, $c_2(x,y) \le 0$ for any $(x,y) \in D_2$;

(11) $p,\ q \in \mathbb{R}$

We have two preliminarly results:

Lemma 1. *If the conditions (5), (6), (8) and*

(12) $h(0)=0,\ h'(x)>0,\ x\in(0,a]$,

(13) $a_1(x,y)=0$ *for any* $(x,y)\in D_1$ *and* $c_2(x,y)=0$ *for any* $(x,y)\in D_2$,

then there exists a function $g \in C^2(\bar{D}\setminus BB_0)$, *such that*

(14) $L_1(g)>0$ *in* $\bar{D}_1\setminus BB_0$, *and*
$L_2(g)>0$ *in* $\bar{D}_2\setminus BB_0$.

Proof. We consider the function g such that

(15) $L_1(g) = g_{xx} + b_1(x,y)g_y + c_1(x,y)g > 0$ in $\bar{D}_1\setminus BB_0$

and

$$(16) \qquad L_2(g) = g_{xx} - h^2(x)\, g_{yy} + q\frac{h'(x)}{h(x)}\, g_x + p h'(x)\, g_y > 0 \quad in \quad \bar{D}_2\setminus BB_0$$

Let be $g: [-a,a] \to \mathbb{R}$, $g \in \mathbf{C}^2([-a,0) \cup (0,a])$, therefore we consider that g is a function of a single variable x.
Then the relation (15) and (16) becomes

$$g'' + c_1(x,y)\, g > 0$$

and

$$g'' + q\frac{h'(x)}{h(x)}\, g' > 0$$

in

$$\bar{D}_2\setminus BB_0$$

When $q \le 0$ we can choice the function g such that $g(x) < 0$ in

$[-a,0)$, respectively $g'(x) < 0$ in $(0,a]$, and $g''(x) > 0$ in $[-a,0) \cup (0,a]$. Thus, for the function $g:[-a,a] \to \mathbb{R}$ given by $g(x) = x^2 - 3a|x|$ we have $g(x) < 0$ in $[-a,0)$, $g'(x)=2x-3a < 0$ in $(0,a]$ and $g''(x) = 2 > 0$ in $[-a,0) \cup (0,a]$.
Therefore, using the relations (6),(12) and the fact that $q \le 0$ we get that the relations (17) and (18) holds.

When $q > 0$ we can choice the function $g\colon [-a,a] \to \mathbb{R}$ defined by

$$g(x)=\begin{cases}\int_x^{-a}(h(-s))^{2q}ds, & x\in[-a,0)\\ 0, & x=0\\ \int_x^{a}(h(s))^{-2q}ds, & x\in(0,a]\end{cases}$$

Thus, for $x \in [-a,0)$ the relation (17) become

$$2q\,h'(-x)\;h^{2q-1}(-x)+c_1(x,y)\int_x^{-a}(h(-s))^{2q}ds>0,$$

which is obvoius fulfilled, again for $x \in (0,a]$ the relation (18) become

$$\frac{2q\,h'(x)}{h^{2q+1}(x)}-q\frac{h'(x)}{h(x)}\cdot\frac{1}{h^{2q}(x)}>0,$$

which is, also, obvious fulfilled.

In conclusion, for the function g considerated above the relation (14) hold, therefore the lemma is proved.

Lemma 2 [10]. *If the conditions (5), (6), (8), (9), (10) holds and*

(19) $a_1(x,y) \le 0$ *for any* $(x,y)\in D_1$.

then there exists a function $g\in C^2(\bar{D}\setminus BB_0)$ *such that*

(20) $L_1(g)<0$ *in* $\bar{D}_1\setminus BB_0$, $L_2(g)<0$ *in* $\bar{D}_2\setminus BB_0$

and

(21) $g_x(x,y)>0$ *in* $\bar{D}_1\setminus BB_0$, $g_x(x,y)<0$ *in* $\bar{D}_2\setminus BB_0$.

Definition 1. We say that the operator L_2 has the maximum property (M.P.) if the conditions (10)

(22) $u \in C^2(\bar{D}_2 \setminus BB_0) \cap C^1(\bar{D}_2)$

(23) $L_2(u) \le 0$ in $\bar{D}_2 \setminus BB_0$

(24) $u_x(0,y) < 0$

and

$$\max_{m_1} u<0 \quad \text{if} \quad c_2\neq0$$

implie that

$$\max_{\bar{D}_1} u = \max_{m_1} u$$

Remark 1. The relation (25) is not necessary if $c_2 = 0$. We have

Theorem 1. *We suppose that the coefficients of the operators* L_1 *and* L_2 *satisfie the conditions (5), (6), (7), (10), (12) and*

(27) $p - q - 1 \geq 0,\ p + q + 1 \leq 0.$

If $u \in C^2(\bar{D}\backslash BB_0) \cap C^1(\bar{D})$ *is a solution of inequations*

(28) $L_1(u)\leq0$ *in* D_1 *and*

(29) $L_2(u)\leq0$ *in* D_2*,*

for which

(30) $u_y(x,0)\leq0,\ u_x(0,y)<0$ *and*

$$(31) \qquad \max_{\bar{D}} u\leq0 \quad \text{when} \quad C_1\neq0$$

and

$$C_2\neq0$$

then we have

$$(32) \qquad \max_{\bar{D}} u=\max_{m_1 \cup B_0A} u$$

Remarks: 2. In the conditions of the Theorem 1 the operator L_2 has the (M.P.) in D_2.

3. If $c_1 = 0,\ c_2 = 0$ the condition (31) is not necessary.

Proof. In the conditions (5), (6), (28) the continuous functions u in D_1 has it negative maximum on the segments AB, BB_0, A_0A. Really the function (-u) satisfie the inequation $L_1(-u) \geq 0$ in D_1 and from [4] it positive maximum is attains on the mentioned segments. Now, we prove that for function u any point $(x_0,0)$ of the segment AB cannot be a negative maximum point. From (28), through past to the limit as $y \to +0$, because of continuity of the derivatives u_x, u_y, u_{xx}, we get that

(33) $u_{xx}(x,0) + a_1(x,0)\, u_{x(x,o)} + {}_{c}1(x,0)\, u(x,0) +$
$+ b_1(x,0)\, u_y(x,0) \le 0.$

therefore

(34) $(-u_{xx}(x,0)) + a_1(x,0)\,(-u_x(x,0)) + c_1(x,0)\,(-u(x,0)) + b_1(x,0)\,(-u_y(x,0) \ge 0.$

From this inequality, because of the relations (5), (6) in a positive maximum $(x_o,0)$ of $(-u)$, must we have

(35) $u_y(x_o,0) > 0.$

Really, in the positive maximum point $(x_o,0)$ for $(-u)$ we have $(-u_{xx}(x_o,0)) < 0$, $(-u_x(x_o,0) = 0$, $-u(x_o,0) > 0$, and how $c_1(x_o,0) < 0$ and $b_1(x_o,0) < 0$, because we have the relation (34) we must have $(-u_y(x_o,0)) < 0$, i.e. (35).

But, the relation (35) is a contradiction with the condition (30). This prove that the function u have not in D_1 a negative maximum attains on the segment AB. It remain then that negative $\overset{max\,u}{\bar{D}_1}$ is attains on yhe segments BB_o, A_oA.

For the domain D_2, if the $\overset{max\,u}{\bar{D}_2}$ is attains in a point $Q \in \bar{D}_2 \setminus BB_o$ we apply Theorem 3 from [7] and so we get a contradiction with the hypotesis of the Theorem 1. Therefore $\overset{max\,u}{\bar{D}_2} = \overset{max\,u}{BB_o}$, and so the Theorem 1 is proved.

Analogously we have

Theorem 2. *In the conditions of the Theorem 1 for the solution of the inequations (28) and (29) we have the relation*

$$\max_{\bar{D}_1} D_+u = \max_{BB_0} u_x = \max_{\bar{D}_1} D_-u$$

where the operators D_+ and D_- *are defined by*

$$D_\pm u = \frac{\sigma u}{\sigma x} \pm h(x)\,\frac{\sigma u}{\sigma y}$$

Proof. We use the Theorem 3[7].

Remarks: On the base of Theorem 1 we can obtain some theorems regarding the uniqueness of the solution for certain boundary value problem for the mixed operator L in the domain D.

5. Using some results of the paper [7] we can obtain a strong estimation for the solution u of the inequations (28) and (29) in which appear and the values of x, u, u_x on BB_o and the values of x, L_2u on D_2.

6. For p = 0, q = −2 and c_2 = 0 on obtain an especially case of some operator which has be considerated by L.E. Payne and D. Sather in 1967.

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