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ON THE CONVERGENCE OF A SEQUENCE

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1. Introduction. Our aim is to investigate the convergence of the following sequence:

(1) $x_n = a^{a^{\cdot^{\cdot^{a^b}}}}$, *a occurs n times*, $a>0$, $b\in\mathbb{R}$, $n\geq 1$.

Observing that it may also be defined by the recurrence

(2) $x_{n+1} = f(x_n)$, $n=0,1,2,\ldots$, $x_0=b\in\mathbb{R}$

where $f:\mathbb{R}\to\mathbb{R}$ denotes the exponential function $f(x)=a^x$, it is convenient to use proposition A or its variant B.

Some particular cases solved in a different way can be found for example in [3] ($a=b=\sqrt{2}$) and [4] ($a\in[1/e,1], b\in\mathbb{R}$).

2. Preliminaries. First we give a proposition connected with fixed points of a real continous function.

Proposition P.

(i) If the fixed point set F of a continous function $f:[a,b)\to\mathbb{R}$ ($a\in\mathbb{R}$; $b\in\mathbb{R}$ or $b=+\infty$; $a<b$) is not empty, then $\exists$ minF.

(ii) If the fixed point set F of a continous function $f:(a,b]\to\mathbb{R}$ ($a\in\mathbb{R}$ or $a=-\infty$; $b\in\mathbb{R}$; $a<b$) is not empty, then $\exists$ maxF.

(iii) If the fixed point set F of a continous function $f:[a,b]\to\mathbb{R}$ ($a,b\in\mathbb{R}$; $a<b$) is not empty, then $\exists$ minF and $\exists$ maxF.

Proof.

(i) Set $F=\{x\in[a,b)\,|\,f(x)=x\}\subset[a,b)$ being bounded below, there exists $c=\inf F\in\mathbb{R}$. In order to show that $c=\min F$, it is enough to verify $c\in F$. We have $c\geq a$ and $c\leq x\in F\subset[a,b)$, so $c\in[a,b)$. Consequently, f is continous at c and $f(x)\to f(c)$ when $x\to c$.

Since $c=\inf F$, given any $\varepsilon>0$ there is an element $x_\varepsilon\in F$ such that $c\le x_\varepsilon=f(x_\varepsilon)<c+\varepsilon$. Letting $\varepsilon\to 0$ we get $x_\varepsilon\to c$ and $f(x_\varepsilon)\to c$. Therefore $f(x)\to c$ when $x\to c$. We conclude $f(c)=c$ i.e. $c\in F$.

(ii) Similar with (i).

(iii) It is the consequence of (i) and (ii).

Using P we can formulate the following propositions.

Proposition A.

Sequence

(3) $x_0\in\mathbb{R}$, $x_{n+1}=f(x_n)$, $n=0,1,2,\ldots$ defined by means of a continous and increasing function $f:\mathbb{R}\to\mathbb{R}$ converges if and only if

(i) $f(x_0)=x_0$ (i.e. $x_0\in F$, F denoting the fixed point set of f)

or (ii) $f(x_0)>x_0$ and $\exists x: x\in F, x>x_0$

or (iii) $f(x_0)<x_0$ and $\exists x: x\in F, x<x_0$.

In case (i) $x_n\to x_0$.

In case (ii) $x_n\to\min\{x\in F\,|\,x>x_0\}$.

In case (iii) $x_n\to\max\{x\in F\,|\,x<x_0\}$.

The sequence (3) is divergent if and only if

(iv) $f(x_0)>x_0$ and $\nexists x: x\in F, x>x_0$ (in this case $x_n\to+\infty$)

or (v) $f(x_0)<x_0$ and $\nexists x: x\in F, x<x_0$ (in this case $x_n\to-\infty$).

Proof. Since f is continous, if the sequence (3) converges, then its limit is a fixed point for f.

(i) If $f(x_0)=x_0$ we have $x_n=x_0$ for any $n=0,1,2,\ldots$.

(ii) In case of $f(x_0)>x_0$, the sequence (3) is increasing. If in addition $\exists x: x\in F, x>x_0$, then the sequence (3) is bounded above by $L=\min\{x\in F\,|\,x>x_0\}$ whose existence results from P(i) applied to $f|_{[x_0,+\infty)}$. (Indeed : $x_0<L$; assuming $x_k<L$ and using the monotonity of f we obtain $x_{k+1}=f(x_k)\le f(L)=L$). Consequently the sequence (3) converges. Denoting $\lim x_n$ by c we have $x_0<f(x_0)=x_1\le c\le L$. Since the open interval (x_0,L) contains no fixed points of f, $c=L$ is necessary.

(iv) If $f(x_0)>x_0$ and $\nexists x: x\in F, x>x_0$, the increasing sequence (3) is unbounded above, so $x_n\to+\infty$. (Supposing sequence (3) bounded above, it is convergent to a limit c which satisfies $x_0<f(x_0)=x_1\le c$; c being a fixed point of f we have a contradiction.)

(iii),(v) If $f(x_0)<x_0$, the sequence (3) is decreasing. Using P(ii) the proof is similar.

Proposition B (being a consequence of A).

Consider the sequence (3) defined by means of a continous and increasing function f. Let $d:\mathbb{R}\to\mathbb{R}$ be the function $d(x)=f(x)-x$ and $D=\{x\in\mathbb{R}\,|\,d(x)=0\}$.

(i) If $d(x_0)=x_0$, then $x_n\to x_0$.

(ii) If $d(x_0)>x_0$ and $\exists x: x\in D, x>x_0$, then $x_n\to \min\{x\in D\,|\,x>x_0\}$.

(iii) If $d(x_0)>x_0$ and $\nexists x: x\in D, x>x_0$, then $x_n\to+\infty$.

(iv) If $d(x_0)<x_0$ and $\exists x: x\in D, x<x_0$, then $x_n\to \max\{x\in D\,|\,x<x_0\}$.

(v) If $d(x_0)<x_0$ and $\nexists x: x\in D, x<x_0$, then $x_n\to-\infty$.

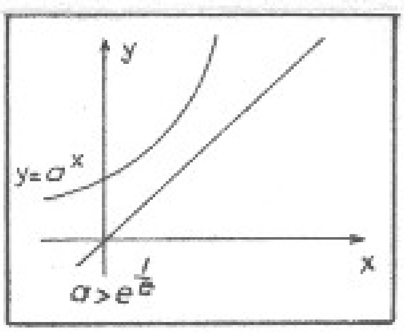
Remark 2.1. Proposition A and B caracterize the started values x_0 which make sequence (3) convergent.

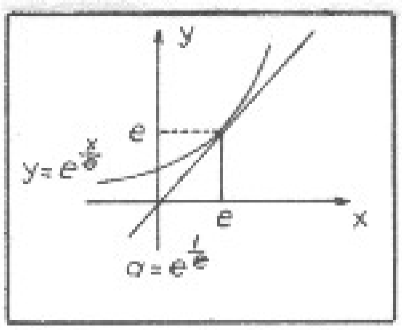
Proposition A needs only the graph of the function f (as compared to the first bisectrix) to be known (including the fixed points of f). Proposition B needs only the sign tabel of d to be known (including the zeros of d).

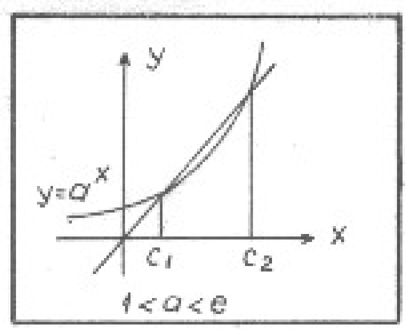
Remark 2.2. If the function f in (3) is decreasing, then $f\circ f$ is increasing, so we can apply proposition A (or B) separately to the partial sequences (x_{2n}) and (x_{2n+1}). In this case f having a unique fixed point c, the sequence (3) is convergent if and only if $x_{2n}\to c$ and $x_{2n+1}\to c$.

3. Now we return to sequence (1)(i.e.(2)).

3.1. If $a>1$, f is strictly increasing. In order to apply proposition A, we must consider three cases depending on the number of fixed points belonging to f (see [2], p.357).







a) If $a>e^{1/e}$, then f has no fixed points and according to A(iv) ve have $x_n\to+\infty$ for any $x_0=b\in\mathbb{R}$.

b) If $a=e^{1/e}$, then f has the unique fixed point $x=e$. Using A(ii) and A(iv), for $x_0=b\le e$ we obtain $x_n\to e$ while for $x_0=b>e$ we get $x_n\to+\infty$.

c) If $a\in(1,e^{1/e})$, then f has two fixed points: $c_1<c_2$.

Applying again proposition A, it follows that

if $x_0=b<c_2$, then $x_n \to c_1$

if $x_0=b=c_2$, then $x_n \to c_2$

if $x_0=b>c_2$, then $x_n \to +\infty$.

Remark. These fixed points are separated by the zeros

$$-\frac{\ln\ln a}{\ln a},\ \frac{1}{\ln a},\ e$$

of the derivatives belonging to the functions a^x-x, $x\ln a-\ln x$, $\ln x+\ln\ln a-\ln\ln x$, which result from the equivalent equations $a^x=x$, $x\ln a=\ln x$, $\ln x+\ln\ln a=\ln\ln x$.

We also have $c_1=a^{c_1}>0 \Rightarrow c_1=a^{c_1}>a^0=1 \Rightarrow c_1=a^{c_1}>a^1=a$, so

$$1<a<c_1<e<\frac{1}{\ln a}<-\frac{\ln\ln a}{\ln a}<c_2$$

and now we can give some started values for $x_0=b$ which in case c) make sequence (1) convergent:

$$\text{any } b<0,\ b=0,\ b\in(0,1),\ b=1,\ b=a,\ b=e^{\frac{1}{e}},\ b=e,\ b=\frac{1}{\ln a},\ b=-\frac{\ln\ln a}{\ln a}\ .$$

3.2. If $a=1$, then for any $b\in\mathbb{R}$ we have $x_1=x_2=\ldots=1$ and $x_n\to 1$.

3.3. If $a\in(0,1)$ then f is strictly decreasing, thus f has a unique fixed point, denoted by c. Therefore, if sequence (x_n) converges, then $x_n\to c$.

It is easy to see that $c\in(0,1)$, moreover $c\in(a,1)$:

$a^x-x\,|_{x=0}=1>0 \quad a^x-x\,|_{x=a}=a^a-a^1>0 \quad a^x-x\,|_{x=1}=a-1<0$.

We shall investigate separately the convergence of partial sequences (x_{2n}) and (x_{2n+1}), applying proposition B to the function $(f\circ f)(x)=a^{a^x}$. This is strictly increasing, has c as a fixed point, and can also have other fixed points. We need the table of signs for the function

$$d(x)=a^{a^x}-x\ .$$

We know that $d(c)=0$. We have

$$d(x)=0 \Leftrightarrow a^{a^x}=x \Rightarrow x>0,\ \text{so}\ d(x)=0 \Leftrightarrow a^x\ln a=\ln x \Leftrightarrow \ln a-\frac{\ln x}{a^x}=0\ .$$

Since $a^x\ln a<0$, it follows that $\ln x<0$ and $x<1$. Consequently, any solution of the equation $d(x)=0$ belongs to $(0,1)$

We shall study the function

$$g(x)=\ln a-\frac{\ln x}{a^{x}} \text{ on } (0,1):\ g'(x)=\frac{x\ln x\ln a-1}{xa^{x}};\ g'(x)=0 \Leftrightarrow x\ln x=\frac{1}{\ln a} \text{ on } (0,1).$$

The following tabel

x	0		$\frac{1}{e}$		1
$h'(x)=1+\ln x$		− − − −	0	+ + + +	
$h(x)=x\ln x$	0	↘	$-\frac{1}{e}$	↗	0

and the relations

$$\frac{1}{\ln a}\in[-\frac{1}{e},0)\Leftrightarrow \ln a\in(-\infty,-e)\Leftrightarrow a\in(0,e^{-e}]$$

show us, that the equation $x\ln x=1/\ln a \Leftrightarrow g'(x)=0$

for $e^{-e}<a<1$ has no solutions;

for $a=e^{-e}$ has the unique solution $x=1/e$;

for $0<a<e^{-e}$ has two solutions ξ_1,ξ_2 verifying $0<\xi_1<1/e<\xi_2<1$.

We need the following limits:

$$\lim_{x\to 0} g(x)=\lim_{x\to 0}(\ln a-\frac{\ln x}{a^{x}})=\ln a-\frac{-\infty}{1}=+\infty;\ g(1)=\ln a<0$$

$$\lim_{x\to 0} g'(x)=\lim_{x\to 0}\frac{x\ln x\ln a-1}{xa^{x}}=\frac{0\ln a-1}{0\cdot 1}=-\infty;\ g'(1)=-\frac{1}{a}<0.$$

a) If $e^{-e}<a<1$, then g' doesn't change its sign on the interval $(0,1)$, so $g'<0$ and the function g is strictly decreasing on $(0,1)$; since

$$\lim_{x\to 0} g(x)>0$$

and $g(1)<0$, it follows that there exists a unique $c_1\in(0,1)$ verifying $g(c_1)=0 \Leftrightarrow d(c_1)=0$. We conclude $c_1=c$.

The functions d and g have the same sign on $(0,1)$; the continous function d has no zeros on $(-\infty,0]\cup[1,+\infty)$, so we can complete the following table:

x	$-\infty$	0	c	1	$+\infty$
g'(x)	(shaded)	– – – – – – –			
g(x)	(shaded)	$+\infty$ ↘	0 ↘	ln a<0	
d(x)	+ + + + + + +		0 – – – – – – –		

Using the proposition B, for any $x_0=b\in\mathbb{R}$ we get that the sequences (x_{2n}) and (x_{2n+1}) are convergent to the limit c. Therefore $\lim x_n=c$ for any $x_0=b\in\mathbb{R}$.

Remark.

We have $c>1/e$. Indeed, assuming $c\le 1/e$ it follows

$$c=a^c\ge a^{\frac{1}{e}}>(e^{-e})^{\frac{1}{e}}=e^{-1}=\frac{1}{e},$$

contradiction.

b) If $a=e^{-e}$, our conclusion is the same as in a). More exactly now $c=1/e$ i.e. $x_n\to 1/e$ for any $x_0=b\in\mathbb{R}$.

c) If $0<a<e^{-e}$, we have

$$g'\left(\frac{1}{e}\right)=\frac{\frac{1}{e}\ln\frac{1}{e}\ln a-1}{\frac{1}{e}\cdot a^{\frac{1}{e}}}=\frac{-\ln a-e}{a^{\frac{1}{e}}}>\frac{e-e}{a^{\frac{1}{e}}}=0\ .$$

The information related to g' lead us to the following table:

x	0	ξ_1	$\frac{1}{e}$	ξ_2	1
g'(x)	–	0	+	0	–
g(x)	$+\infty$ ↘		↗	↘	ln a<0

We know that c is a fixed point of g. We'll show that $c\in(\xi_1,\xi_2)$. It is enough for this (see the table above) to show that $g'(c)>0$. First we notice that $c<1/e$ (assuming $c\ge 1/e$ it follows

$$c=a^c\le a^{\frac{1}{e}}<(e^{-e})^{\frac{1}{e}}=\frac{1}{e},$$

contradiction).

Then the numerator of $g'(c)$: $c\ln c\ln a-1=\ln c\ln a^{c}-1=\ln c\ln c-1>0$, so $g'(c)>0$. Since $c\in(\xi_1,\xi_2)$, $g(c)=0$ and g increase on (ξ_1,ξ_2), we get $g(\xi_1)<0$, $g(\xi_2)>0$. According to the last table, it follows that g (and d) has three zeros on (0,1):

$c_1\in(0,\xi_1)$, $c\in(\xi_1,\xi_2)$, $c_2\in(\xi_2,1)$.

Since on (o,1) the functions d and g have the same sign and the continous function d hasn't any zero on $(-\infty,0]\cup[1,\infty)$, we get the following table:

x	$-\infty \quad 0 \quad c_1 \quad \xi_1 \quad c \quad \frac{1}{e} \quad \xi_2 \quad c_2 \quad 1 \quad +\infty$
$g(x)$	$+\infty \quad + \; 0 \; - \; - \; - \quad 0 \; + \; + \; + \; + \quad 0 \; - \quad \ln a<0$
$d(x)$	$+ \; + \; + \; + \quad 0 \; - \; - \; - \quad 0 \; + \; + \; + \; + \quad 0 \; - \; - \; -$

Applying proposition B, for $x_0=b\in(-\infty,c)$ we get $x_{2n}\to c_1$ and for $x_0=b\in(c,+\infty)$ we get $x_{2n}\to c_2$.

In both cases $\lim x_{2n}\neq c$, thus $\lim x_n$ does not exist.

It can be noticed that for $x_0=b\in(-\infty,c)$ (because of $x_1=a^{x_0}>a^{c}=c$) we have $x_{2n+1}\to c_2$, while for $x_0\in(c,+\infty)$ (because of $x_1<c$) we have $x_{2n+1}\to c_1$.

If $x_0=b=c$, then obviously $x_1=x_2=\ldots=c$, so $\lim x_n=c$.

Our final conclusion:

Cases depending *on* a and b		Behaviour of (x_n)
$e^{1/e}<a$	$b\in\mathbb{R}$	$x_n\to+\infty$
$a=e^{1/e}$	$b>e$ $b\leq e$	$x_n\to+\infty$ $x_n\to e$
$1<a<e^{1/e}$	$b>c_2$ $b=c_2$ $b<c_2$	$x_n\to+\infty$ $x_n\to c_2$ $x_n\to c_1$
$a=1$	$b\in\mathbb{R}$	$x_n=1\to1$
$e^{-e}<a<1$	$b\in\mathbb{R}$	$x_n\to c$
$a=e^{-e}$	$b\in\mathbb{R}$	$x_n\to1/e$
$0<a<e^{-e}$	$b\in(c,+\infty)$ $b=c$ $b\in(-\infty,c)$	$x_{2n}\to c_2$, $x_{2n+1}\to c_1$ $x_n=c\to c$ $x_{2n}\to c_1$, $x_{2n+1}\to c_2$

Remark 3.1. The sequence (1) is convergent for any $x_0=b\in\mathbb{R}$ if and only if $a\in[e^{-e},1]$.

Remark 3.2. The sequence

$$x_n=a^{a^{\cdot^{\cdot^{a}}}},\ a\ occurs\ n\ times, a>0,\ n\geq 1$$

which follows from (1) when $x_0=b=a$, is convergent if and only if

$$a\in[e^{-e}, e^{\frac{1}{e}}].$$

Finally we propose for discussion the case $a=b=\alpha^{\frac{1}{\alpha}}$ with $\alpha>0$.

4. In order to study other reccurent sequences, there can be formulated different versions of proposition A (and of its variant B, as well).

Proposition A1. Let $f:[a,+\infty)\to\mathbb{R}$ ($a\in\mathbb{R}$) be a continous and increasing function. Let $F=\{x\in[a,+\infty)\,|\,f(x)=x\}$.
Consider the sequence $x_0\in[a,+\infty)$, $x_{n+1}=f(x_n)$, $n=0,1,2...$.

(i) If $f(x_0)=x_0$, then $x_1=x_2=...=x_0$ and $x_n\to x_0$.

(ii) If $f(x_0)>x_0$ and $\exists x:x\in F, x>x_0$, then $x_n\to\min\{x\in F\,|\,x>x_0\}$.

(iii) If $f(x_0)>x_0$ and $\nexists x:x\in F, x>x_0$, then $x_n\to+\infty$.

(iv) If $f(x_0)<x_0$ and $\exists x:x\in F, x<x_0$, then $x_n\to\max\{x\in F\,|\,x<x_0\}$.

(v) If $f(x_0)<x_0$ and $\nexists x:x\in F, x<x_0$, then the sequence $(x_n)_{n\geq 0}$ can not be defined.

Proposition A2. Let $f:(-\infty,b)\to\mathbb{R}$ ($b\in\mathbb{R}$) be a continous and increasing function. Let $F=\{x\in(-\infty,b)\,|\,f(x)=x\}$.
Consider the sequence $x_0\in(-\infty,b)$, $x_{n+1}=f(x_n)$, $n=0,1,2...$.

(i) If $f(x_0)=x_0$ then $x_1=x_2=...=x_0$ and $x_n\to x_0$.

(ii) If $f(x_0)>x_0$ and ($\exists x:x\in F, x>x_0$ or $f(x)\to b$ when $x\nearrow b$), then $x_n\to\min\ \{x\in F\,|\,x>x_0\}\cup\{b\}$.

(iii) If $f(x_0)>x_0$ and $\nexists x:x\in F, x>x_0$ and $f(x)\not\to b$ when $x\nearrow b$, then the sequence $(x_n)_{n\geq 0}$ can not be defined.

(iv) If $f(x_0)<x_0$ and $\exists x:x\in F, x<x_0$, then $x_n\to\max\{x\in F\,|\,x<x_0\}$.

(v) If $f(x_0)<x_0$ and $\nexists x:x\in F, x<x_0$, then $x_n\to-\infty$.

Proposition A3. Let $f:(a,b]\to\mathbb{R}$ ($a,b\in\mathbb{R}$) be a continous and increasing function. Let $F=\{x\in(a,b]\,|\,f(x)=x\}$.
Consider the sequence $x_0\in(a,b]$, $x_{n+1}=f(x_n)$, $n=0,1,2,...$.

(i) If $f(x_0)=x_0$ then $x_1=x_2=...=x_0$ and $x_n\to x_0$.

(ii) If $f(x_0)>x_0$ and $\exists x:x\in F$, $x>x_0$, then $x_n\to\min\{x\in F\,|\,x>x_0\}$.

(iii) If $f(x_0)>x_0$ and $\nexists x:x\in F, x>x_0$, then the sequence $(x_n)_{n\geq 0}$ can not be defined.

(iv) If $f(x_o)<x_o$ and $(\exists x: x\in F, x<x_o$ or $f(x)\to a$ when $x\searrow a)$, then $x_n \to \max \{a\} \cup \{x\in F \mid x<x_o\}$.

(v) If $f(x_o)<x_o$ and $\nexists x: x\in F, x<x_o$ and $f(x)\not\to a$ when $x\searrow a$, then the sequence $(x_n)_{n\geq o}$ can not be defined.

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