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A REMARKABLE POINT ON THE BISECTRIX OF A TRIANGLE

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In this paper we will use the known notations for any triangle ABC. The sim of this work is distinguishes a very important point on the bisectrix of a triangle, point that has very intresting and original properties as it can be scen next.

To demonstrate a some theorems in this paper we'll use the results obtained in other works:

In [1] and other publications is given the next theorem:

Theorem 1: *In an triangle ABC the middle point of a side, the middle of the adegnate height and Lemoine K point (the symediane centre) are colinear.*

The line established by these points is called Schlömilch line.

Note: Any triangle ABC has three Schlömilch lines. Schlömilch's line which contains the middle of (BC) will be marked with d_a: d_b and b_c being analogons marks.

In the paper [2] is given the next theorem:

Theorem 2: *If on the sides (AB), (AC) of any triangle ABC are taken the points, M, N, the line MN passes throngh the Lemoine K point if and only if:*

$$b^2\cdot\frac{MB}{MA}+c^2\cdot\frac{NC}{NA}=a^2$$

Then we'll obtain the results:

Theorem 3: *If AD is an arbitrary cevian (Ceva's line) of a triangle ABC and on the halflines (AB, (AC are taken the points E, F such as $\sphericalangle EDA \equiv \sphericalangle B$, $\sphericalangle FDA \equiv \sphericalangle C$ and if $\{P\} = AD \cap EF$ then there* *is the relation:*

$$AD^2 = BD \cdot DC \cdot \frac{AP}{PD}$$

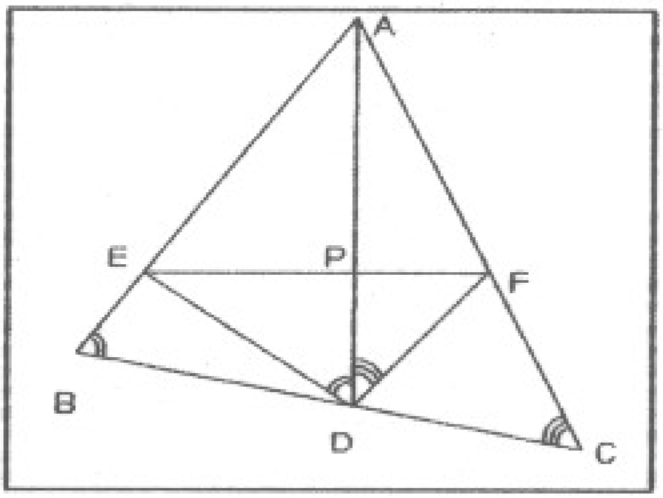
Proof: The quadrangle AEDF being inscriptible ($\sphericalangle EDF + \sphericalangle A = \sphericalangle B + \sphericalangle C + \sphericalangle A = 180^\circ$) it results that:

$$\triangle AEP \sim \triangle FDP$$

and

$$\triangle AFP \sim \triangle EDP$$

From the similitude of these triangles rezult the relations:



$$\frac{AE}{DF} = \frac{AP}{PF}$$

and

$$\frac{AF}{DE} = \frac{PF}{PD}$$

If we multiply member by member we obtain

$$\frac{AE}{F} \cdot \frac{AF}{DE} = \frac{AP}{PD} \qquad (1)$$

On the other hand from the similitude of the triangles:

$$\triangle ADE \sim \triangle ABD$$

and

$$\triangle ADF \sim \triangle ACD$$

results the relation:

$$\frac{AE}{DE} = \frac{AD}{BD} \qquad (2)$$

and

$$\frac{AP}{DP}=\frac{AD}{DC} \qquad (3)$$

Ont of (2) and (3) relation result

$$AD^2=BD\cdot DC\cdot\frac{AP}{PD}$$

Corollary 3.1 If, in the conditions of the third theorem, AD is a bisectrix then

$$\frac{AP}{PD}=\frac{4p(p-a)}{a^2}$$

Proof: From the third theorem and from the bisectrix theorem result that

$$AD^2=\frac{(a^2bc)}{(b+c)^2}\cdot\frac{AP}{PD} \qquad (1)$$

Ont of relation (1) and ont of relations:

$$AD=\frac{2bc\cdot\cos\frac{A}{2}}{b+c}$$

and

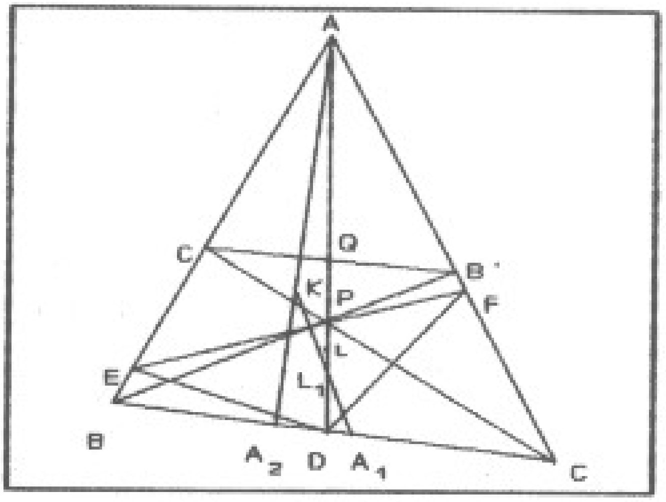
$$\cos^2\cdot\frac{A}{2}=\frac{p(p-a)}{bc}$$

We obtain

$$\frac{AP}{PD}=\frac{4p(p-a)}{a^2}$$

Theorem 4: *If AD is the bisectrix of any triangle ABC and on the halfline , (AB, (AC are taken the points E, F such as* $\sphericalangle EDA = \sphericalangle B$, $\sphericalangle FDA = \sphericalangle C$ *and if the intersection are considerend:* $\{P\} = AD\cap EF$, $\{B'\}= AC\cap BP$, $\{C'\} = AB\cap CP$ *and* $\{Q\} = AD\cap C'B'$, *then the middle L of segment (QD) there is on the Schlömilch line* d_a.

Proof:



Without restricting the generality we may suppose that AB<AC ($c<b$). Let A_1 be the middle of (BC) and $A_2\in(BC)$ the foot of the symediane descended from A. Of course the Lemoine K point is on the symediane AA_2. Schlömilch's line d_a coincides with KA_1 and this line cuts AD, because AA_1 and AA_2 are izogonal cevians A_1 and A_2 are on one side and the otoer side of D point on the BC line. Let $\{L_1\} = AD\cap KA_1$ and suppose $L\neq L_1$. Applying Menelau's theorem on the A_2AD triangle with the transversal KA_1 results that

$$\frac{A_2K}{AK}\cdot\frac{AL_1}{L_1D}\cdot\frac{A_1D}{A_1A_2}=1$$

of that we infer the relation

$$\frac{AL_1}{L_1D}=\frac{AK}{A_2K}\cdot\frac{A_1A_2}{A_1D} \qquad (1)$$

Because L is Lemoine's point of the ABC triangle it results

$$\frac{AK}{A_2K}=\frac{b^2+c^2}{a^2} \qquad (2)$$

On the other hand from the relations:

$$A_1A_2=BA_1-BA_2\ ,\ BA_1=\frac{a}{2}$$

and

$$BA_2=\frac{ac^2}{b+c}$$

we infer that

$$A_1A_2=\frac{a(b^2-c^2)}{2(b^2+c^2)} \qquad (3)$$

Of the relation

$$A_1D=BA_1-BD\ ,BA_1=\frac{a}{2}$$

and

$$BD=\frac{ac}{b+c}$$

we obtain

$$A_1D=\frac{a(b-c)}{2(b+c)} \qquad (4)$$

Of the relations: (1),(2), (3) and (4) results that

$$\frac{AL_1}{L_1D}=\frac{(b+c)^2}{a^2} \qquad (5)$$

Of the relation (5) we infer

$$\frac{AL_1}{L_1D}-1=\frac{(b+c)^2-a^2}{a^2}$$

Of this results the relation

$$\frac{AL_1}{L_1D}-1=\frac{4p(p-a)}{a_2} \qquad (6)$$

Of the corollary 3.1 and the relation (6) we obtain

$$\frac{AL_1}{L_1D}-1=\frac{AP}{PD} \qquad (7)$$

Of this results the relation

$$\frac{AL_1}{L_1D}=\frac{AP}{PD}+1$$

Cevians AD, BB' and CC' being junctured in P results that

$$\frac{AP}{PD}=\frac{2AQ}{QD} \qquad (8)$$

Of the relations (7) and (8) results that

$$\frac{AL_1}{L_1D}=\frac{2AQ}{QD}+1$$

Where from adding at each member 1 we infer that

$$\frac{AD}{L_1D}=\frac{2AD}{QD}$$

Of the last relation results $2L_1D = QD$, from this results that L_1 is the middle of (QD), this is a contradiction, a segment cannot have two middles so L is the middle of (QD), that is $L \in d_m$.

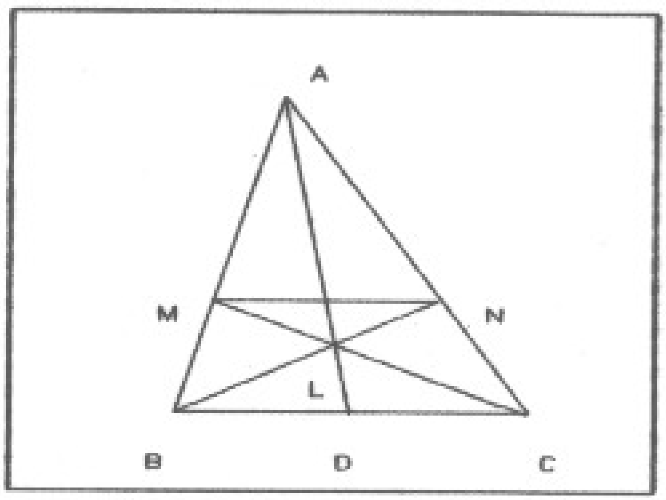
With the help of theorem 4 we can enunciate the next remarkable result.

Theorem 5: *The middle of the side (BC), the middle of the segment (QD), the middle of the height desended from A and Lemoine K are colinear (on the same line).*

An other remarkable result of L point is given by the next theorem:

Theorem 6: *If the point L is the one determined throngh theorem 4 and if {M} = AB∩CL and {N} = AC∩BL then Lemoine K point is on MN.*

Proof:



Of Menelaus's theorem applied to ABD triangle with the transversal MC we infer the relation

$$\frac{MB}{MA}=\frac{LD}{AL}\cdot\frac{BC}{DC} \qquad (1)$$

Of relation (1) and of the relations

$$\frac{DC}{BC}=\frac{b}{b+c}\ ,\ \frac{AL}{LD}=\frac{(b+c)^2}{a^2}$$

(see relation (5) of the proof of the theorem 4) results that

$$\frac{MB}{MA}=\frac{a^2}{b(b+c)} \qquad (2)$$

On the same way deduce the relation

$$\frac{NC}{NA}=\frac{a^2}{c(b+c)} \qquad (3)$$

Of the relations (2) and (3) we infer that

$$b^2\cdot\frac{MB}{MA}+c^2\frac{NC}{NA}=\frac{a^2b}{b+c}+\frac{a^2c}{b+c}=a^2$$

or

$$b^2 \cdot \frac{MB}{MA} + c^2 \frac{NC}{NA} = a^2$$

this relation expresses $K \in MN$ as stated by theorem 2.

Theorem 7: *In any triangle ABC be the bisectrix AD and on the halflines (AB, (AC be the points E, F so that ∢EDA = ∢B and ∢FDA = ∢C. We consider the intersections: (P) = AD∩EF, (B') = AC∩BP, (C') = AB∩CP and (Q) = AD∩C'B'. Be L the middle of (QD) and be (M) = AB∩CL and (N) = AC∩BL. In these conditions Schlömilch's line* d_a *cuts MN in Lemoine K point.*

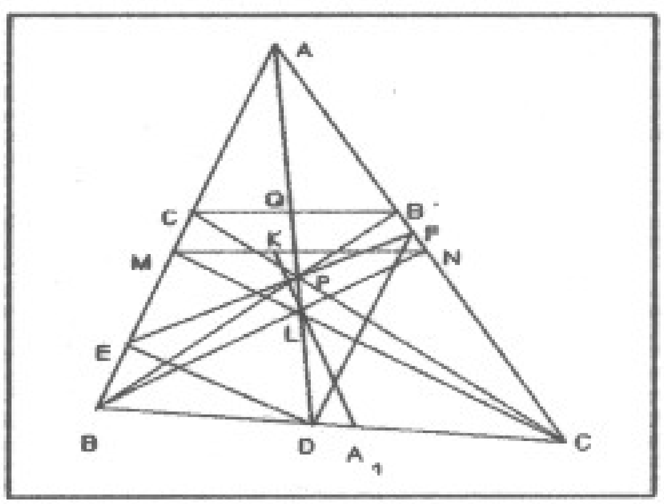
Proof: As $K \in d_a$ and stated by theorem 6 $K \in MN$ results that $K = d_a \cap MN$.

Note: Whith the help of these theorems can be given more original constructions for Lemoine point (symedian centre) withont appbying at symedianes or circumscribed circle. More in these constructions it is used, at the most, the most, the middle of a side of the triangle ABC or, at the most, the middle of a height.

As it follows I'll give three constructions for Lemoine's point.

Construction 1.

If A_1 is the middle of (BC) then $(K) = MN \cap A_1L$.
This construction rezults from $K \in MN$ and $A_1L = d_a$



Construction 2. If H_1 is the middle of the height descended from A then $(K) = H_1L \cap MN$.
This constructions results from $K \in MN$ and $d_a = H_1L$. In the end I'll give a construction without using the middles of the sides of the triangle ABC and the middles of the heights.

Construction 3. If the point L obtained ont of the theorem 4 is on the bisectrix descended from A I'll mark this point with L_a and M and N points generated by L we'll be marked with M_a and N_a. As stated by theorem 6 $K \in M_aN_a$. It is evidently that ont the bisectrix descended from B can be con constructed analogically a point L_b that generates a secant M_bN_b. Because $K \in M_aN_a$ and $K \in M_bN_b$ results that $\{K\} = M_aN_a \cap M_bN_b$.

R E F E R E N C E S

1. NICOLESCU, L., BOSKOFF, V.: Probleme practice de geometrie. Editura Tehnica Bucuresti - 1990

2. OPREA, N.: Ceviene de rang k, Gazeta Matematica 8/1989.

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