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ON AN INTEGRAL EQUATION OF VOLTERRA TYPE USING A GENERALIZED LIPSCHITZ CONDITION

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1. Introduction

In applications of the contraction mapping principle to concrete problems an usual way is to associate a fixed point operator equation. The problem which arises is to construct a norm, equivalent to the norms of the spaces into consideration, with respect to which the operator in question is contractive.

In this note we shall give an "existence and uniqueness" theorem for an integral equation of Volterra type using the generalized contraction principle [2], when generalized norms (with values in the positive cone of a real ordered Banach space) are considered.

2. A fixed point theorem in K - metric spaces

We need some definitions and results from [1] - [5], [9] concerning generalized norms, i.e., norms which take values in an abstract cone K, and comparison operators $\varphi:K \to K$, which enjoys certain properties in common with the map $t \to at$, $0<a<1$, but is not necessarily linear.

Let $(E, \|\cdot\|)$ be a real Banach space. A set $K\subset E$ is called a <u>cone</u> if

(i) K is closed;

(ii) $x,y\in K$ implies $ax+by\in K$ for all $a,b\in \mathbb{R}_+$;

(iii) $K \cap(-K)=\{\theta\}$, where θ is the null element of E.

The cone induces a reflexive, transitive and antisimetrical order relation in E, by

$x \le y$ if and only if $y - x \in K$,

related to the linear structure by the properties

$u \le v$ implies $u+z \le v+z$, for each $z \in E$

and

$u \le v$ implies $tu \le tv$, for each $t \in \mathbb{R}_+$,

that is, " $\le$ " is a linear order relation.

The space E endowed with this order relation is called an ordered Banach space, while K is termed as its positive cone.

We say that the norm of E is monotone if $x,y \in E$, $\theta \le x \le y$ implies $\|x\| \le \|y\|$.

The cone K is called normal if there exists $\delta > 0$ such that, from $x,y \in E$, $x,y \ne \theta$ and $\|x\| = \|y\| = 1$ it results $\|x+y\| > \delta$.

Recall that if the norm of E is monotone, then K is a normal cone [6].

Throughout this paper K will be the positive cone in a real ordered Banach space $(E, \|\cdot\|)$ with monotone norm.

Definition 1.

An operator $\varphi: K \to K$ is a comparison operator if

(i) φ is monotone increasing ;

(ii) $(\varphi^n(t))_{n \in \mathbb{N}}$ converges to θ, for all $t \in K$.

Example 1.

Let $E = \mathbb{R}$, the real axis, with the usual norm. In this case $K = \mathbb{R}_+$ and a typical comparison function is $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$,

$$\varphi(t) = at \, , \; 0<a<1 \, , \; t \in \mathbb{R}_+$$

Definition 2.

An operator $\varphi: K \to K$ is called (c) - comparison operator if φ is monotone increasing and fulfils the following convergence condition

(c) There exist two numbers k_0, α, $0<\alpha<1$, and a convergent series with nonnegative real terms $\sum_{k=1}^{\infty} a_k$

$$\varphi^{k+1}(t) \le \alpha\varphi^{k}(t) + a_k, \quad for \quad k \ge k_0 \; , \quad \forall t \in K$$

Remark.

Every (c) - comparison operator is a comparison operator since, if φ is a (c) - comparison operator, then (see [1], [2]) the series $\sum_{k=1}^{\infty} \varphi^{k}(t)$ converges for all $t \in K$ and, consequently, the series

$$\sum_{k=1}^{\infty} \varphi^{k}(t)$$

converges in E, hence condition (ii) in definition 1 is satisfied.

The function φ from example 1 is a (c) - comparison function.

Definition 3.

Let X be a nonempppty set A mapping

d: X x X → K is said to be a K-metric on X if

(i) $d(x,y) \ge \theta$, for all $x,y \in X$ and $d(x,y)=\theta \Leftrightarrow x=y$;

(ii) $d(x,y) = d(y,x)$, for all $x,y \in X$;

(iii) $d(x,y) \le d(x,z) + d(z,y)$, for all $x,y,z \in X$.

The obtained entity: the nonempty set X with a K-metric d is called K-metric space, denoted as usually (X,d).

Definition 4.

Let (X,d) be a K-metric space and φ: K → K a comparison operator. A mapping f: X → X is called φ-contraction if there exist a comparison operator φ: K → K such that

$$d(f(x), f(y)) \le \varphi\, (d(x,y)) \; , \text{ for all } x,y \in X$$

Example 2.

Let $K = \mathbb{R}_+$ and let φ be as in example 1. Then a $\mathbb{R}_+$-metric space is an usual metric space, while condition (2) becomes the wellknown contraction condition.

Remark.

In a K-metric space the concepts as fundamental sequence, convergent sequence and complete K-metric space are defined in similar manner to the usual metric spaces.

As shown by example 2, the following result is a generalization of the contraction mapping principle [2], [3].

THEOREM 1.

Let (X,d) be a complete K-metric space, where K is a normal cone and $f: X \to X$ a φ-contraction where φ is a (c)-comparison operator.

Then

1) $F_f = \{x^*\}$, where $F_f = \{x \in X \,/\, f(x) = x\}$;

2) The sequence $(x_n)_{n\in N}$, $x_n = f(x_{n-1})$, $n \geq 1$, converges to x^*, for every $x_0 \in X$;

3) We have

$$d(x_n, x^*) \leq s(d(x_n, x_{n+1})),$$

where $s(t)$ denote the sum of the series (1);

4) If, in addition, φ is subadditive and there exist $\eta \in K$, $\eta \neq \Theta$ and a mapping $g: X \to X$, so that

$$d(f(x), g(x)) \leq \eta \,, \text{ for all } x \in X,$$

we have

$$d(y_n, x^*) \leq \eta + s(\eta) + s(dx_n, x_{n+1})),$$

where $y_n = g^n(x_0)$.

Remark.

When $K = \mathbb{R}_+$ and φ is an in example 1, from theorem 1 we obtain the contraction mapping principle [9], [10].

3. Integral equations of Volterra type

Let us consider the following Volterra equation

$$u(x,y) = \lambda \int_0^x \int_0^y H(s,t,u(s,t))\,ds\,dt + h(x,y), \quad (x,y) \in D,$$

where $\lambda \in \mathbb{R}, D = [0,a] \times [0,b]$, $H \in C(D \times \mathbb{R})$, $h \in C(D)$. Let X be the space C(D), endowed with the usual norm.

We denote by K the cone of the positive functions from X and we define a mapping $\|\cdot\|_* : X \to \mathbb{R}$, by $\|u\|_* = |u|$, *for each* $u \in X$.

It is obvious that $\|\cdot\|_*$ is a generalized norm on X, that is

(i) $\|u\|_* \geq \Theta$, for each $u \in X$ and $\|u\|_* = 0$ if and only if $u = \Theta$, the null function;

(ii) $\|\lambda u\|_* = |\lambda| \cdot \|u\|_*$, for each $u \in X$ and any $\lambda \in \mathbb{R}$;

(iii) $\|u + v\|_* \leq \|u\|_* + \|v\|_*$, for each $u, v \in X$.

By other hand, K is the positive cone of X endowed with the Cebîşev's norm, which is monotone, hence K is a normal cone.
The partial order induced by K on X is given by

$$u \leq v \Leftrightarrow u(x,y) \leq v(x,y),\ \forall (x,y) \in D.$$

If we consider the mapping $T : X \to X$, defined by

$$T\,u(x,y) = \lambda \int_0^x \int_0^y H(s,t,u(s,t))\,ds\,dt + h(x,y), \quad (x,y) \in D,$$

we conclude that every solution of equation (3) is a fixed point of T and vice versa. Thus we obtain the following result regarding the existance and unicity of the solutions of equation (3).

THEOREM 2.

Assume that

(i) $H \in C(D \times \mathbb{R})$ *and* $h \in C(D)$;

(ii) There exists a function $g: D \times \mathbb{R}_+ \to \mathbb{R}_+$, *integrable with respect to the first argument and monotone increasing with respect to the second argument, i.e,* $g(\cdot, \cdot, u)$ *is integrable on D for fixed* $u \in \mathbb{R}_+$ *and*

$$u \leq v \Rightarrow g(x,y,u) \leq g(x,y,v),\ \forall (x,y) \in D,$$

such that

$$|H(x,y,u) - H(x,y,v)| \leq g(x,y,|u-v|), \text{ for each } (x,y) \in D,\ u,v \in \mathbb{R}_+;$$

(iii) If $\varphi: K \to K$ *is defined by*

$$\varphi\, u(x,y) = |\lambda| \int_0^x \int_0^y g(s,t,u(s,t))\,ds\,dt,$$

then there exist a number α, $0<\alpha<1$, *and a convergent series of nonnegative terms* $\sum_{k=0}^{\infty} a_k$, *such that*

$$\varphi^{k+1}(r) \le \alpha\varphi^k(r) + a_k, \text{ for every } r\in K,\ k\ge k_0 \text{(fixed)}$$

Under these assumptions the equation (3) has a unique solution $\bar{u}$, *which may be obtained by the successive approximation method, starting from an arbitrary element* $u_0\in X$.

The sequence of succesive approximations, $(u_p)_{p\in\mathbb{N}}$, is given by

$$u_p(x,y) = \lambda \int_0^x \int_0^y H(s,t,u_{p-1}(s,t))\,ds\,dt + h(x,y),$$

and we have the following estimation

$$\|u_p - \bar{u}\|_* \le s(\|u_p - u_{p+1}\|_*).$$

Proof.

From condition (4) we deduce that φ *is monotone increasing and from (iii) it results that* φ *is a (c)-comparison operator.*

From (5) we obtain

$$\|Tu - Tv\|_* = |T\,u(x,y) - T\,v(x,y)| =$$

$$= \left| \lambda\int_0^x \int_0^y | H(s,t,u(s,t)) - H(s,t,v(s,t)) |\,ds\,dt \right| \le$$

$$\le |\lambda| \int_0^x \int_0^y | H(s,t,u(s,t)) - H(s,t,v(s,t)) |\,ds\,dt \le$$

$$\le |\lambda| \int_0^x \int_0^y g(s,t,|u(s,t) - v(s,t)|)\,ds\,dt = \varphi(\|u - v\|_*),$$

that is, T is a φ-contraction.

Now theorem 1 completes the proof.

Remark.

If $H(s,t,u(s,t)) = G(s,t)\cdot f(u(s,t))$, we obtain a Volterra integral equation with deviating argument.

For $f(t) = t$, equation (3) is just the Volterra integral equation in the two dimensional case.

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