Buletinul Ştiinţific al Universităţii din Baia Mare
Seria B, Matematică-Informatică, vol.IX(1993), 9-18

ON THE APPROXIMATION OF THREEVARIATE B-CONTINUOUS FUNCTIONS USING BERNSTEIN STANCU TYPE OPERATORS

Dan BĂRBOSU

Abstract: A constructive proof for a uniform approximation theorem of a three variate B-continuous function is presented.

1. Introduction.

Let us denote $I_3 = [0,1]x[0,1]x[0,1]$ and $F(I_3)=\{f|f:I_3 \to \mathbb{R}\}$

1.1. Definition [9]: The operator Δ_3: $F(I_3) \to F(I_3)$, defined by

(1.1) $\Delta_3(f;M,M') = f(x,yz) - f(x',y,z) - f(x,y',z) - f(x,y,z') + + f(x,y',z') + f(x',y,z') + f(x',y',z') - f(x',y',z')$, is called (operator) threevariate difference of $f\in F(I_3)$ on $[M,M']$ where $M(x,y,z)$ and $M'(x',y'z')\in I_3$.

It is easy to see that Δ_3 is a linear operator.

1.2. Definition [9]: The function $f\in F(I_3)$ is called B-continuous on $M(x,y,z)\in I_3$ if:

(1.2) $$\lim_{M'\to M} \Delta_3(f;M,M') = 0$$

If f is B-continuous in every point of I_3 then f is called B-continuous on I_3. We denote by $C_b(I_3) = \{f\in F(I_3)|f.$B-continuous on $I_3\}$ and $C(I_3) = \{f\in F(I_3)|f$-continuous on $I_3\}$.

1.3. Lemma: If $f\in C_b(I_3)$, the function $g\in F(I_3)$ defined by

(1.3) $g(x',y',z') = f(x',y,z) + f(x,y',z) + f(x,y,z') - f(x,y',z')- - f(x',y',z') - f(x',y',z) + f(x',y',z')$ is continuous on I_3.

Proof: Let $M(x,y,z)\in I_3$ fixed and $M(x',y'z')\in I_3$ variable. It is easy to see that $g(x,y,z) = f(x,y,z)$ and $g(x,y,z)-$ $-g(x',y',z') = \Delta_3(f;\ M,M')$. From this relations we have

$$\lim_{M'\to M} [g(x,y,z) - g(x',y',z')] = \lim_{M'\to M} \Delta_3(f;M,M') = 0$$

and therefore g is continuous on every point $M(x,y,z)$ of I_3.

1.4. Definition [9]: The function $f\in F(I_3)$ is called B-bounded on I_3 if we have:

(1.4) $|\Delta_3(f;M,M')| \le K,\ (\forall)M,M'\in I_3$ where $K>0$.

A direct consequence of lemma 1.3 is

1.5. Lemma: If $f\in C_b(I_3)$, then f is B-bounded on I_3.

2. Fundamental polynomials of Bernstein Stancu type.

In this section we will present some concepts and results from [3], [4], [5], [13] which will be used in our paper.

We denote by $a^{[p,\delta]}$ the factorial power of p order and δ parameter of a, i.e. $a^{[p,\delta]} = a(a-\delta)(a-2\delta)\ldots(a-(p-1)\delta)$.

2.1 Definition: If $\alpha=\alpha(m)\ge 0$, $\beta=\beta(n)\ge 0$, $\gamma=\gamma(l)\ge 0$ we denote by $W_{m,i}(x,\alpha)$, $W_{n,j}(y,\beta)$, $W_{l,k}(z,\gamma)$ the polynomials:

(2.1) $$W_{m,i}(x,\alpha) = \binom{m}{i}\frac{x^{[i,-\alpha]}\ (1-x)^{[m-i,-\alpha]}}{1^{[m,-\alpha]}}$$

(2.2) $$W_{n,j}(y,\beta) = \binom{n}{j}\frac{y^{[j,-\beta]}\ (1-y)^{[n-j,-\beta]}}{1^{[n,-\beta]}}$$

(2.3) $$W_{l,k}(z,\gamma) = \binom{l}{k}\frac{z^{[k,-\gamma]}\ (1-z)^{[l-k,\gamma]}}{1^{[l,-\gamma]}}$$

The polynomials (2.1), (2.2) and (2.3) are called fundamental polynomials of Bernstein Stancu type.

The properties of the polynomials (2.1), (2.2), (2.3) are contained in

2.2. Lemma [3], [5]: The fundamental polynomials of Bernstein Stancu type satisfies the inequalities:

$$(2.4)\qquad \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{l}(i-mx)^2 W_{m,j}(x,\alpha)W_{n,j}(y,\beta)W_{l,k}(z,\gamma) \le \frac{m^2}{4(1+\alpha)}\left(\frac{1}{m}+\alpha\right)$$

$$(2.5)\qquad \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{l}(j-ny)^2 W_{m,i}(x,\alpha)W_{n,j}(y,\beta)W_{l,k}(z,\gamma) \le \frac{n^2}{4(1+\beta)}\left(\frac{1}{n}+\beta\right)$$

$$(2.6)\qquad \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{l}(k-lz)^2 W_{m,i}(x,\alpha)W_{n,j}(y,\beta)W_{l,k}(z,\gamma) \le \frac{l^2}{4(1+\gamma)}\left(\frac{1}{l}+\gamma\right)$$

$$(2.7)\qquad \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{l}(i-mx)^2(j-ny)^2 W_{m,i}(x,\alpha)W_{n,j}(y,\beta)W_{l,k}(z,\gamma) \le$$

$$\le \frac{mn}{16(1+\alpha)(1+\beta)}\left(\frac{1}{m}+\alpha\right)\left(\frac{1}{n}+\beta\right)$$

$$(2.8)\qquad \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{l}(i-mx)^2(k-lz)^2 W_{m,i}(x,\alpha)W_{n,j}(y,\beta)W_{l,k}(z,\gamma) \le$$

$$\le \frac{ml}{16(1+\alpha)(1+\gamma)}\left(\frac{1}{m}+\alpha\right)\left(\frac{1}{l}+\gamma\right)$$

$$(2.9)\qquad \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{l}(j-ny)^2(k-lz)^2 W_{m,i}(x,\alpha)W_{n,j}(y,\beta)W_{l,k}(z,\gamma) \le$$

$$\le \frac{nl}{16(1+\beta)(1+\gamma)}\left(\frac{1}{n}+\beta\right)\left(\frac{1}{l}+\gamma\right)$$

$$(2.10)\qquad \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{l}(i-mx)^2(j-ny)^2(k-lz)^2 W_{m,i}(x,\alpha)W_{n,j}(y,\beta)W_{l,k}(z,k) \le$$

$$\le \frac{mnl}{64(1+\alpha)(1+\beta)(1+\gamma)}\left(\frac{1}{m}+\alpha\right)\left(\frac{1}{n}+\beta\right)\left(\frac{1}{l}+\gamma\right)$$

3. The uniform approximation of the threevariate B-continuous functions

Let $f \in F(I_3)$. We denote by $P_m^{[x,\alpha]}f$, $P_n^{[y,\beta]}f$, $P_l^{[z,\gamma]}f$ the parametric extensions of the univariate Bernstein Stancu type operators, defined by:

$$(3.1) \qquad (P_m^{[x,\alpha]}f)(x,y,z) = \sum_{i=0}^{m} W_{m,i}(x,\alpha)\, f(\frac{i}{m},y,z)$$

$$(3.2) \qquad (P_n^{[y,\beta]}f)(x,y,z) = \sum_{j=0}^{n} W_{n,j}(y,\beta)\, f(x,\frac{j}{n},z)$$

$$(3.3) \qquad (P_l^{[z,\gamma]}f)(x,y,z) = \sum_{k=0}^{l} W_{l,k}(z,\gamma)\, f(x,y,\frac{k}{l})$$

3.1. Definition. The boolean sum of the parametric extensions (3.1), (3.2), (3.3) denoted by $B_{m,n,l}^{[\alpha,\beta,\gamma]}f = P_m^{[x,\alpha]} \oplus P_n^{[y,\beta]} \oplus P_l^{[z,\gamma]}$ is defined by:

$$(3.4) \qquad B_{m,n,l}^{[\alpha,\beta,\gamma]} = P_m^{[x,\alpha]} + P_n^{[y,\beta]} + P_l^{[z,\gamma]} - P_m^{[x,\alpha]}P_n^{[y,\beta]} - P_m^{[x,\alpha]}P_l^{[z,\gamma]} - P_n^{[y,\beta]}P_l^{[z,\gamma]} + P_m^{[x,\alpha]}P_n^{[y,\beta]}P_l^{[z,\gamma]}$$

The operator (3.4) associated to the function $f \in F(I_3)$ the pseudopolynomial in Marchaud sense [8] defined by:

$$(3.5) \qquad (B_{m,n,l}^{[\alpha,\beta,\gamma]}f)(x,y,z) = \sum_{i=0}^{m} W_{m,i}(x,\alpha)\,W_{n,j}(y,\beta)\,W_{l,k}(z,\gamma)\,|\, f(\frac{i}{m},y,z) + f(x,\frac{j}{n},z) + f(x,y,\frac{k}{l}) - f(\frac{i}{m},\frac{j}{n},z) - f(\frac{i}{m},y,\frac{k}{l}) - f(x,\frac{j}{n},\frac{k}{l}) + f(\frac{i}{m},\frac{j}{n},\frac{k}{l})\,|$$

The main result of this note is:

3.2. Theorem: *If*:

i) $f \in C_b(I_3)$

ii) $\alpha=\alpha(m)\to 0 \ (m\to\infty)$, $\beta=\beta(n)\to 0 \ (n\to\infty)$, $\gamma=\gamma(l)\to 0 \ (l\to\infty)$

then the row $\{ B_{m,n,l}^{[\alpha,\beta,\gamma]} f \}_{m,n,l\in N}$ *, with the general term given in (3.5)*

is convergent to f, uniformly on I_3.

Proof. We consider the approximation formula

$$(3.6) \quad f = B_{m,n,l}^{[\alpha,\beta,\gamma]} f + R_{m,n,l}^{[\alpha,\beta,\gamma]} f$$

and we will show that $R_{m,n,l}^{[\alpha,\beta,\gamma]} \to 0$, uniformly on I_3.

From (3.5) and from the identity

$\sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{l} W_{m,i}(x,\alpha) \cdot W_{n,j}(y,\beta) W_{l,k}(z,\gamma) = 1$ proved in [13], we obtain:

$$(3.7) \quad | f(x,y,z) - (B_{m,n,l}^{[\alpha,\beta,\gamma]} f)(x,y,z) | \leq$$

$$\leq \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{l} W_{m,i}(x,\alpha) \ W_{n,j}(y,\beta) \ W_{l,k}(z,\gamma) \cdot$$

$$\cdot \ |\Delta_3(f; (x,y,z), (\frac{i}{m}, \frac{j}{n}, \frac{k}{l}))|$$

Let $(\forall)\, \varepsilon 0$. Because $f \in C_b(I_3)$, in the points of I_3 in which we have:

$$(3.8) \quad |x-\frac{i}{m}| < \delta, \ |y-\frac{j}{n}| < \delta, \ |z-\frac{k}{l}| < \delta \text{ the inequality}$$

$$|\Delta_3(f; (x,y,z), (\frac{i}{m}, \frac{j}{n}, \frac{k}{l}))| < \frac{\varepsilon}{8} \text{ is true.}$$

We denote by:

S_1 - the sum of the terms from (3.7) in which the inequalities (3.8) are valid

S_2 - the sum of the terms from (3.7) in which only the first two inequalities (3.8) are valid

S_3 - the sum of the terms from (3.7) in which only the first and the third inequalities (3.8) are valid

S_4 - the sum of the terms from (3.7) in which only the second and the third inequalities (3.8) are valid

S_5 - the sum of the terms from (3.7) in which only the first inequality (3.8) is valid

S_6 - the sum of the terms from (3.7) in which only the second inequality (3.8) is valid

S_7 - the sum of the terms from (3.7) in which only the third inequality (3.8) is valid

S_8 - the sum from the terms from (3.7) in which the three inequalities (3.8) are not valid

Using the sums $S_i(i=\overline{1,8})$, we can write the inequality (3.7) under the form:

$$(3.9) \quad |\, f(x,y,z) - (B_{m,n,l}^{[\alpha,\beta,\gamma]} f)(x,y,z)\,| \le \sum_{i=1}^{8} S_i$$

In S_1 we have $|x-\frac{i}{m}|<\delta,\ |y-\frac{j}{n}|<\delta,\ |z-\frac{k}{l}|<\delta$ and from here it results

$|\,\Delta_3(f;\ (x,y,z),(\frac{i}{m},\frac{j}{n},\frac{k}{l}))\,|<\frac{\varepsilon}{8}$. We obtain the inequality:

$$(3.10) \quad S_1<\frac{\varepsilon}{8}\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{l} W_{m,i}(x,\alpha)\ W_{n,j}(y,\beta)\ W_{l,k}(z,\gamma) = \frac{\varepsilon}{8}$$

In S_2 we have $|z-\frac{k}{l}| \geq \delta$, from which we obtain:

$$\frac{(k-lz)^2}{l^2} \geq \delta^2 \Rightarrow \frac{(k-lz)^2}{l^2\delta^2} \geq 1 \Rightarrow$$

$$\Rightarrow |\Delta_3(f;(x,y,z),(\frac{i}{m},\frac{j}{n},\frac{k}{l}))| \leq \frac{(k-lz)^2}{l^2\delta^2} \cdot |\Delta_3(f;(x,y,z),(\frac{i}{m},\frac{j}{n},\frac{k}{l}))$$

Using the lemma 1.5, the function f is B-bounded on I_3.

We denote by: $K = \sup_{(x,y,z)\in I_3} \{ |\Delta_3(f;(x,y,z),(\frac{i}{m},\frac{j}{n},\frac{k}{l}))|$

$| i=\overline{0,m},\ j=\overline{0,n},\ k=\overline{0,l},\ |z-\frac{k}{l}|\geq\delta,\ |x-\frac{i}{m}|<\delta,\ |y-\frac{j}{n}|<\delta \}$. Using the inequality (1.10), we obtain:

(3.11) $\quad S_2 \leq \frac{K}{4\delta^2(1+\gamma)}(\frac{1}{l}+\gamma)$

Taking account of the relation $\lim_{l\to\infty}\gamma(l) = 0$, from (3.11) we obtain:

(3.12) $\quad S_2 < \frac{\varepsilon}{8}$

Similarly we can prove the inequalities $S_i<\frac{\varepsilon}{8}\,(i=\overline{3,8})$.

We conclude that for $(\forall)\ \varepsilon>0$, the inequality:

$|f(x,y,z) - (B_{m,n,l}^{[\alpha,\beta,\gamma]} f)(x,y,z)|<\varepsilon,\ (\forall)\ (x,y,z)\in I_3$ hold. This inequality shows that $R_{m,n,l}^{[\alpha,\beta,\gamma]} f\to 0$ and the theorem is proved.

3.3 Remarks

i) For $\alpha=\beta=\gamma=0$ we obtain the result of [9].

ii) From the lemma 1.5 it results that the B-boundary condition from [9] is not necessary.

iii) The result of the theorem 3.2 is true on every compact of $\mathbb{R}^3$.

R E F E R E N C E S

1. BADEA, I., Asupra unei teoreme de aproximare uniformă prin pseudopolinoame de tip Bernstein, Analele Univ. Craiova, seria a V-a, nr. 2-1974, pag. 55-58

2. BADEA, I., Modulul de continuitate în sens Bögel şi unele aplicatii, Studia Universitetis Babes Bolyai, Series Mathematica - mechanica, fasciculus 2(1973), pag. 70-78.

3. BĂRBOSU, D., L'approximation par des polynômes du type Bernstein Stancu des fonctions Bögel continues, Bul.St. al Univ. Baia Mare, seria B, Mat. - Inf, Vol. VIII (1992), pag. 11-18.

4. BĂRBOSU, D., Considerations sur l'approximation des fonctions de deux variables par des produits tensoriels et sommes booléens d'operateurs du type Bernstein Stancu (to appear).

5. BĂRBOSU, D., Aplicaţii ale teoriei lui W.J.Gordon în aproximarea functiilor de trei variabile (to appear).

6. COMAN, Gh., Analiză numerică, lito. Univ. Babes Bolyai, Cluj Napoca 1980, pag. 187-209.

7. GORDON, W.J., Distributive Latties and the Approximations of Multivariate Functions, in "Approximation with Special Emphasis on Spline Functions", Ed. I. J. Schoenberg, Academic Press, New York, pag.223-277.

8. MARCHAUD, A., Sur dérivées et les differences des fonctions de variables reéles, J. de Math. 1927, vol. 6, pag. 337-425

9. MUREŞAN, T., Polinoame de tip Bernstein pentru funcţii tridimensional continue, lucrările seminarului "Theodor Angheluţă" lito. Inst.Polit. Cluj Napoca, 1983, pag.247-251

10. MUREŞAN, T., Contribuţii în analiza numerică globală (teză de doctorat), Cluj Napoca, 1984

11. NICOLESCU, M., Analiză matematică II, Ed.tehnică 1958, pag. 425-436

12. POPOVICIU, T., Sur quelques propriétés des fonctions d'une ou de deux variables réelles, Mathematica 1934, pag. 1-85

13. STANCU, D.D., Aproximarea funcţiilor de două şi mai multe variabile prin operatori de tip Bernstein, Studii şi cercetări matematice, Tome 22, nr.2(1970), pag. 335-345

UNIVERSITY OF BAIA MARE
DEPARTMENT OF MATHEMATICS
4800 BAIA MARE
ROMÂNIA