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ON SOME MEAN VALUE THEOREMS FOR REAL FUNCTIONS OF VECTORIAL VARIABLE

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In this paper we consider a number of generalizations in $\mathbb{R}^n$ of the classical mean value theorems for real-valued functions of one real variable and some of their consequences with geometric interpretation.

Throughout this paper D; $D\subseteq\mathbb{R}^n$ denotes a conex domain, $\langle , \rangle$ the canonic scalar product of two vectors from $\mathbb{R}^n$, $\| \ \|$ the euclidean norm in $\mathbb{R}^n$ and for $a,b\in\mathbb{R}^n$, $[a,b]=\{x\in\mathbb{R}^n;\ x=a+t(b-a);\ t\in[0,1]\}$.

THEOREM 1 *Let $f,g: D \to \mathbb{R}$ be continuous functions on $[a,b]$; $a,b\in D$ such that the gradients ∇f and ∇g exists for all $x\in]a,b[$. Then there is at least one $c\in]a,b[$ such that*

$$\langle (f(b)-f(a))\nabla g(c) - (g(b)-g(a))\nabla f(c),\ b-a\rangle = 0 \tag{1}$$

PROOF. We consider the function $F: [0,1] \to \mathbb{R}$ given by

$$F(t) = (f(b)-f(a))((g\circ x)(t)-g(a)) - (g(b)-g(a))((f\circ x)(t)-f(a))$$

where $x: [0,1] \to [a,b];\ x(t) = a+t(b-a)$.

Obviously $F(0)=F(1)=0$, F is a continuous function on [0,1] and $F'(t)$ exists for all $t\in]0,1[$, hence there is at least one $\zeta\in]0,1[$ for which $F'(\zeta)=0$. According to the rule of the composed functions derivation we have

$$F'(\zeta) = (f(b)-f(a))\langle \nabla g(c),\ b-a\rangle-(g(b)-g(a))\ \langle \nabla f(c),b-a\rangle = 0$$

where $c=a+\zeta(b-a)$, and this gives the result (1).

COROLLARY 1 (The generalization of Cauchy's mean value theorem). *Let $f,g: D \to \mathbb{R}$ be continuous functions on $[a,b]$, $a,b \in D$ such that ∇f, ∇g exist for each $x \in]a,b[$ and $\langle \nabla g(x), b-a \rangle \neq 0$. Then $g(a) \neq g(b)$ and there is at least one $c \in]a,b[$ such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{\langle \nabla f(c), b-a \rangle}{\langle \nabla g(c), b-a \rangle} \tag{2}$$

This follows from the theorem 1 and Rolle's theorem applied to the function $G: [0,1] \to \mathbb{R}$; $G = g \circ x$ where $x: [0,1] \to [a,b]$; $x(t) = a+t(b-a)$. Indeed, $g(a)=g(b)$, therefore $G(0)=G(1)$ implies that $\exists \zeta \in]0,1[$; $G'(\zeta)=0$ and that contradicts the hypothesis $\langle \nabla g(x), b-a \rangle \neq 0$ $\forall x \in]a,b[$.

COROLLARY 2 (Lagrange's mean value theorem). *Let $f: D \to \mathbb{R}$ be a continuous function on $[a,b] \subseteq D$ such that for each $x \in]a,b[$, $\nabla f(x)$ exists.*

Then there is at least one $c \in]a,b[$ such that

$$f(b)-f(a) = \langle \nabla f(c), b-a \rangle \tag{3}$$

This follows from the corollary 1 for $g(x) = \langle x, b-a \rangle$, because $\nabla g = b-a$ and $g(b)-g(a) = \| b-a \|^2$.

Remarks 1. The results of theorem 1 do not implie

$$(f(b)-f(a))\nabla g(c) = (g(b)-g(a))\nabla f(c) .$$

For example if $f,g: \mathbb{R}^2 \to \mathbb{R}$; $f(x_1,x_2) = x_1+2x_2$; $g(x_1,x_2) = x_1-x_2$ then

$(g(b)-g(a))\nabla f(x) \neq (f(b)-f(a)) \nabla g(x)$ for all $x \in \mathbb{R}^2$ if $a \neq b$;

2. The relation (1) can be written

$$(f(b)-f(a))\,\delta(f(c),b-a) = (g(b)-g(a))\,\delta(g(c),b-a) \tag{1'}$$

where $\delta(h(c),b-a)$ denotes the directional derivative of f at c on the direction from a through b.

The geometrical interpretation of Corollary 2:

The continuous function $\psi: [0,1] \to D \times \mathbb{R}$ defined by

$t \longrightarrow (a+t(b-a), f(a+t(b-a))$ can be regarded as a (simple) path in $D \times \mathbb{R}$([1]). Because $\psi'(t)$ exists and $\psi'(t) = (b-a, \langle \nabla f(x(t)), b-a \rangle) \neq 0$

for each $t\in]0,1[$, the path ψ has a tangent at each point other than endpoints $A = \psi(0) = (a, f(a))$; $B=\psi(1) = (b, f(b))$, with direction $\frac{\psi'(t)}{\|\psi'(t)\|}$. Corollary 2 asserts that there is at least one

intermediate point $(c,f(c)$ of the path ψ at which the direction of the tangent is the same as that of the chord AB, or, equivalently, there is at least one point of the graph of f at which the normal $(\nabla f(c),-1)$ to this is orthogonal with the chord AB.

THEOREM 2 *Let $f: D \to \mathbb{R}$ be a continuous function on $[a,b]\subset D$, such that the gradient ∇f exists for all $x\in]a,b[$. Then for each $d=a+t(b-a)$; $t\in\mathbb{R}\setminus[0,1]$, there is $c\in]a,b[$ such that*

$$\frac{\langle af(b)-bf(a)-d(f(b)-f(a)),b-a\rangle}{\|b-a\|^2} = \langle \nabla f(c), c-d\rangle - f(c) \tag{4}$$

PROOF. The functions $f^*, g: [a,b] \to \mathbb{R}$

$f^*(x) = \frac{f(x)}{\langle x-d, b-a\rangle}$; $g(x) = \frac{1}{\langle x-d, b-a\rangle}$ are continuous on $[a,b]$ and

differentiable on $]a,b[$. Because $\nabla g(x) = -\frac{b-a}{\langle x-d, b-a\rangle^2} \neq 0$, by corollary 1 there is at least one $c\in]a,b[$ such that

$$\frac{\langle a-d,b-a\rangle f(b) - \langle b-d,b-a\rangle f(a)}{\langle a-d,b-a\rangle - \langle b-d,b-a\rangle} = -\frac{\langle \nabla f(c)\langle c-d,b-a\rangle-(b-a)f(c),b-a\rangle}{\|b-a\|^2}.$$

By the properties of the scalar product and the colinearity of c-d and b-a we have:

$$\langle \nabla f(c), b-a\rangle \langle c-d, b-a\rangle = \langle \nabla f(c), c-d\rangle \|b-a\|^2$$

and therefore this gives (4).

Remark: In particular, the case $n = 1$ gives the Rotaru's theorem [5].

The geometrical interpretation. The theorem 2 asserts that there is at least one intermediate point $(c,f(c))$, $c\in]a,b[$, so that the straight line (∇) in $\mathbb{R}^{n+1}$ which focuses the points $A=(a,f(a))$, $B=(b,f(b))$ of the graph of f, the straight line "verticale", $x=d$, $d=a+t(b-a)$; $t\in\mathbb{R}\setminus[0,1]$ and the tangent hyperplane at $(c,f(c))$ at the graph of f are concurrent.

COROLLARY (The generalization of Pompeiu's theorem [3]). *Let $f: D \to \mathbb{R}$ be a continuous function $a,b \in D$ such that $b=\lambda a$, $\lambda>0$ and ∇f exists for each $x \in]a,b[$.*
Then there is at least one $c \in]a,b[$ such that

$$\frac{\langle af(b)-bf(a), b-a\rangle}{\|b-a\|^2} = \langle \nabla f(c), c\rangle - f(c) \tag{5}$$

THEOREM 3 *Let $f: D \to \mathbb{R}$ be a continuous function on $[a,b] \subset D$, differentiable at $]a,b[$ and such that $\langle \nabla f(x), b-a\rangle \neq 0$, $\forall x \in]a,b[$ (that implie $f(a) \neq f(b)$). For each $\mu \in \mathbb{R} \setminus Imf_{[a,b]}$; $Imf_{[a,b]} = \{f(x) / x \in [a,b]\}$ there is $c \in]a,b[$ such that*

$$\frac{\langle af(b)-bf(a), b-a\rangle}{(f(b)-f(a))\ \|b-a\|^2} + \frac{\mu}{f(b)-f(a)} = \frac{\langle c, b-a\rangle}{\|b-a\|^2} - \frac{f(c)-\mu}{\langle \nabla f(c), b-a\rangle} \tag{6}$$

PROOF. The functions $f^*, g: [a,b] \to \mathbb{R}$;

$f^*(x) = \dfrac{\langle x, b-a\rangle}{f(x)-\mu}$; $g(x) = \dfrac{1}{f(x)-\mu}$ are continuous on $[a,b]$ and ∇f^*, ∇g exist for each $x \in]a,b[$.

Because $\langle \nabla g(x), b-a\rangle = -\dfrac{\langle \nabla f(x), b-a\rangle}{(f(x)-\mu)^2} \neq 0$, by Corollary 1 of Theorem 1 we have (6).

COROLLARY *Let $f: D \to \mathbb{R}$ be a continuous function on $[a,b] \subset D$ such that $f(x) \neq 0$ $\forall x \in [a,b]$ and $\nabla f(x)$ exist $\langle \nabla f(x), b-a\rangle \neq 0$, $\forall x \in]a,b[$. Then $f(a) \neq f(b)$ and there is $c \in]a,b[$ such that*

$$\frac{\langle af(b)-bf(a), b-a\rangle}{f(b)-f(a)} = \langle c, b-a\rangle - \frac{f(c)}{\langle \nabla f(c), b-a\rangle} \|b-a\|^2 \tag{7}$$

In particular, the case n=1 gives the Ivan's theorem [3].

The geometrical interpretation of theorem 3.

The relation (6) can be written

$$\frac{\langle a-c, b-a\rangle}{\|b-a\|^2} + \frac{\mu-f(a)}{f(b)-f(a)} + \frac{f(c)-\mu}{\langle \nabla f(a), b-a\rangle} = 0 \tag{6'}$$

Because $\langle a-c,b-a\rangle\langle\nabla f(c),b-a = \langle a-c,\nabla f(c)\rangle \| b-a\|^2$, from (6') we have

$$\left\langle a+\frac{\mu-f(a)}{f(b)-f(a)}(b-a)-c,\ \nabla f(c)\right\rangle-(\mu-f(c))=0 \tag{6''}$$

This shows that there is at least one point $(c,f(c))$ of the graph of f such that the tangent hyperplane in this point

$$\langle x-c,\nabla f(c)\rangle-(x_{n+1}-f(c))=0\ ,$$

the straight line which focuses the points $(a,f(a))$, $(b,f(b))$ of the graph of f and the hyperplane $x_{n+1}=\mu$ are concurrent.

In the sequel weill prove a very interesting property of the intermediar point defined by value theorem 1.

THEOREM 4 *Let $f,g: D\to\mathbb{R}$ be the functions of class $C^2(D)$, let $a\in D$ such that $\langle\nabla g(x),x-a\rangle\neq 0$ for all $x\in D\setminus\{a\}$ and $d^2f(a)dg(a)\neq$ $\neq d^2g(a)df(a)$. By theorem 1, for all $x\in D\setminus\{a\}$ there is at least one point $c_x\in]a,x[$ such that*

$$(f(x)-f(a))\langle\nabla g(c_x),x-a\rangle=(g(x)-g(a))\langle\nabla f(c_x),x-a\rangle$$

and

$$\lim_{x\to a}\frac{\|c_x-a\|}{\|x-a\|}=\frac{1}{2} \tag{8}$$

PROOF Let $e=(e_1,e_2,\ldots,e_n)$ be the unit vector in $\mathbb{R}^n$. Because D is a convex domain in $\mathbb{R}^n$, for all $x\in D$ such that $x-a$ is collinear with e, there is the compact interval $I\subseteq\mathbb{R}$ such that $x=a+t_xe$; $t_x\in I$. Obviously $c=a+t_ce$; $|t_c|<|t_x|$; $t_ct_x>0$.

The function $F: I\to\mathbb{R}$; $F(t)=(f(a+te)-f(a))\langle\nabla g(a),e\rangle-$ $-(g(a+te)-g(a))\langle\nabla f(a),e\rangle$ is of class $C^2(I)$ and

$$F'(t)=\langle\nabla f(a+te),e\rangle\langle\nabla g(a),e\rangle-\langle\nabla g(a+te),e\rangle\langle\nabla f(a),e\rangle$$

$$F''(t)={}^tEH_f(a+te)E\langle\nabla g(a),e\rangle-{}^tEH_g(a+te)E\langle\nabla f(a),e\rangle$$

where $H_f(a+te)$; $H_g(a+te]$ are the Hesse-matrices of f respectively g in the point $a+te$ and $E={}^T(e_1,\ldots,e_n]\in\mathcal{M}_{n,1}(\mathbb{R})$.

Because F(0)=F'(0)=0, by l'Hôspital theorem we have

$$\lim_{t\to 0}\frac{F(t)}{t^2} = \lim_{t\to 0}\frac{F'(t)}{2t} = \frac{1}{2}\lim_{t\to 0}F''(t) = \frac{1}{2}F''(0) \tag{9}$$

But, $F(t) = (f(a+te)-f(a))\langle\nabla g(a)-\nabla g(c),e\rangle +$

$$+ (f(a+te)-f(a))\langle\nabla g(c),e\rangle - (g(a+te)-g(a))\langle\nabla f(a),e\rangle$$

and by theorem 1:

$$F(t) = (f(a+te)-f(a))\langle\nabla g(a)-\nabla g(c),e\rangle +$$

$$+ (g(a+te)-g(a))\langle\nabla f(c)-\nabla f(a),e\rangle .$$

Since

$$\lim_{t\to 0}\frac{F(t)}{t^2} = \lim_{t\to 0}\frac{f(a+te)-f(a)}{t}\ \lim_{t\to 0}\langle\frac{\nabla g(a)-\nabla g(c)}{t},e\rangle +$$

$$+ \lim_{t\to 0}\frac{g(a+te)-g(a)}{t}\ \lim_{t\to 0}\langle\frac{\nabla f(c)-\nabla f(a)}{t},e\rangle =$$

$$= -\langle\nabla f(a),e\rangle\,\langle\lim_{t_c\to 0}\frac{\nabla g(c)-\nabla g(a)}{t_c},e\rangle\ \lim_{t\to 0}\frac{t_c}{t} +$$

$$+ \langle\nabla g(a),e\rangle\,\langle\lim_{t_c\to 0}\frac{\nabla f(c)-\nabla g(a)}{t_c},e\rangle\ \lim_{t\to 0}\frac{t_c}{t} =$$

$$= \lim_{t\to 0}\frac{t_c}{t}\,({}^tEH_f(a)E\,\langle\nabla g(a),e\rangle - {}^tEH_g(a)E\,\langle\nabla f(a),e\rangle) .$$

As $t_c t_x > 0$ we have $\frac{t_c}{t_x} = \frac{\|c-a\|}{\|x-a\|}$ and therefore

$$\lim_{t\to 0}\frac{F(t)}{t^2} = \lim_{x\to a}\frac{\|c-a\|}{\|x-a\|}\cdot F''(0) \tag{10}$$

From (10) and (9) and because of the hypothesis F"(0)≠0, we have

$\lim_{t\to 0}\frac{\|c-a\|}{\|x-a\|}=\frac{1}{2}$ for all directions e, that is $\lim_{x\to a}\frac{\|c_x-a\|}{\|x-a\|}=\frac{1}{2}$.

Remark The theorem 4 generalizes the Popa's result [4].

ABSTRACT

In this paper, the generalization of the Cauchy's mean value theorem for real-valued function of vectorial variable in $\mathbb{R}^n$ and some of their consequences with geometric interpretation are considered. We also prove an interesting property of the intermediar point defined in the mean value theorem 1.

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