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SOME HERMITE-BIRKHOFF SPLINE INTERPOLATION

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0. Summary. It is considered an interpolating Hermite-Birkhoff problem of a given function by polynomial spline of degree 2m+1. The spline interpolation is constructed such that it interpolates the derivatives of function up to the order m at the knots of a uniform partition and the values of function at the ends.

Following particular results obtained in [2,3] the even-degree case was treated in [1].

Construction, existence, and uniqueness are given for the spline interpolation function.

1. Statement of the problem. Let Δ_n be a uniform partition of the interval $[a,b]$, i.e. the knots are given by $x_k=a+kh$, $k=\overline{0,n+1}$, $h=(b-a)/(n+1)$.

We denote $\mathcal{S}_{2m+1}(\Delta_n)$ the space of polynomial splines of degree 2m+1 with deficiency m in each knot of Δ_n.

Therefore $s \in \mathcal{S}_{2m+1}(\Delta_n)$ if and only if $s \in C^{m+1}[a,b]$ and its restriction to any subinterval $\left[x_k,x_{k+1}\right]$ is a polynomial of degree 2m+1.

Let consider the problem of approximating a function f on $[a,b]$ by a spline function $s_f \in \mathcal{S}_{2m+1}(\Delta_n)$ such that

$$s_f(x_0)=f_0;\ s_f^{(i)}(x_k)=f_k^{(i)}\ ,\ k=\overline{0,n+1},\ i=\overline{1,m};\ s_f(x_{n+1})=f_{n+1},$$

when $f_0=f(x_0)$, $f_k^{(i)}=f^{(i)}(x_k)$, $f_{n+1}=f(x_{n+1})$ are given.

2. Two-point Hermite-interpolating polynomial. There exists explicit formula for the Hermite interpolating polynomial of degree p+q+1 which matches g and its first p derivatives at the node α, and also g and its first g derivatives at the node $\beta\neq\alpha$ [4]. Namely, it has expression

$$H_{p+q+1}(g;x) = \sum_{i=0}^{p} h_{\alpha,i}(x)g^{(i)}(\alpha) + \sum_{i=0}^{q} h_{\beta,j}(x)g^{(j)}(\beta),$$

where the fundamental Hermite interpolating polynomials are given by

$$h_{\alpha,i}(x) = \frac{(x-\alpha)^i}{i!}\left(\frac{x-\beta}{\alpha-\beta}\right)^{q+1}\sum_{\nu=0}^{p-i}\binom{q+\nu}{\nu}\left(\frac{x-\alpha}{\beta-\alpha}\right)^{\nu},\ i=\overline{0,p},$$

and

$$h_{\beta,j}(x) = \frac{(x-\beta)^j}{j!}\left(\frac{x-\alpha}{\beta-\alpha}\right)^{p+1}\sum_{\mu=0}^{q-j}\binom{p+\mu}{\mu}\left(\frac{x-\beta}{\alpha-\beta}\right)^{\mu},\ j=\overline{0,q}.$$

Lemma 1. If $p=q=m$, $\alpha=x_k$, $\beta=x_{k+1}$ and setting $x=x_k+th$ $(0\le t\le 1)$ then

$$(1)\qquad h_{x_k,i}(x) = \frac{h^i t^i (1-t)^{m+1}}{i!}\sum_{\nu=0}^{m-i}\binom{m+\nu}{\nu}t^{\nu},\ i=\overline{0,m},$$

and

$$(2)\qquad h_{x_{k+1},j}(x) = \frac{h^j (t-1)^j\ t^{m+1}}{j!}\sum_{\mu=0}^{m-j}\binom{m+\mu}{\mu}\left(1-t\right)^{\mu},\ j=\overline{0,m}.$$

Lemma 2. If $p=q=m$, $\alpha=0$, $\beta=1$, then

$$(3)\qquad h_{0,i}(t) = A_{m,i}(t) = \frac{t^i (1-t)^{m+1}}{i!}\sum_{\nu=0}^{m-i}\binom{m+\nu}{\nu}t^{\nu},\ i=\overline{0,m},$$

and

$$(4)\qquad h_{0,j}(t) = B_{m,j}(t) = \frac{(t-1)^j\ t^{m+1}}{j!}\sum_{\mu=0}^{m-j}\binom{m+\mu}{\mu}\left(1-t\right)^{\mu},\ j=\overline{0,m}.$$

Lemma 3. Following relations are satisfied

$$h_{x_k,i}(x) = h^i A_{m,i}(t),\ i=\overline{0,m}\ ,$$

and

$$h_{x_{k+1},j}(x) = h^j B_{m,j}(t),\ j=\overline{0,m}\ .$$

The three lemmas are immediately obtained as particular cases of the fundamental Hermite interpolating polynomials.

Proposition 4. For polynomials $A_{m,i}(t)$ and $B_{m,j}(t)$ we have following formulas

$$(5)\qquad A_{m,i}(t) = \frac{t^i (1-t)^{m+1}}{i!}\sum_{\nu=0}^{m-i} a_m^{(\nu)} t^{\nu},\ i=\overline{0,m}$$

$$(6)\qquad B_{m,j}(t) = \frac{(t-1)^j\ t^{m+1}}{j!}\sum_{\mu=0}^{m-j} b_{m,j}^{(\mu)} t^{\mu},\ j=\overline{0,m},$$

where the coefficients $a_m^{(\nu)}$ and $b_{m,j}^{(\mu)}$ are given by recurrence formulas

$$\begin{cases} a_m^{(o)} = 1 \quad , \quad a_1^{(1)} = 2\ , \\ a_m^{(\nu)} = a_{m-1}^{(\nu)} + a_m^{(\nu-1)}\ , \quad \nu=\overline{1,m-1}\ , \\ a_m^{(m)} = a_m^{(m-1)} + \sum_{\nu=0}^{m-1} a_{m-1}^{(\nu)}\ ,\ m\geq 2\ , \end{cases}$$

and

$$\begin{cases} b_{m,m}^{(o)} = 1\ ,\quad b_{m,o}^{(m)} = (-1)^m a_m^{(m)}\ , \\ b_{m,o}^{(\mu-1)} = -\frac{\mu(m+\mu+1)}{(m+\mu)(m-\mu+1)}\, b_{m,o}^{(\mu)}\ ,\ \mu=m,m-1,\ldots,1, \\ b_{m,j}^{(\mu)} = \frac{m-j-\mu+1}{2m-j+2}\, b_{m,j-1}^{(\mu)}\ ,\ j=\overline{1,m}\ ,\quad \mu=\overline{0,m-j}. \end{cases}$$

<u>Proof</u>. The first group of relations is obtained taking into account that $a_m^{(\nu)} = \binom{m+\nu}{\nu}$.

For the second group of relations we have successively

$$B_{m,j}(t) = \frac{(t-1)^j\ t^{m+1}}{j!}\sum_{\mu=0}^{m-j}\binom{m+\mu}{\mu}\left[\sum_{\nu=o}^{\mu}\binom{\mu}{\nu}\left(-1\right)^{\nu} t^{\nu}\right] =$$

$$= \frac{(t-1)^j\ t^{m+1}}{j!}\sum_{\mu=0}^{m-j}\left(-1\right)^{\mu}\left[\sum_{\nu=\mu}^{m-j}\binom{m+\nu}{\nu}\binom{\nu}{\mu}\right]t^{\mu} =$$

$$= \frac{(t-1)^j\ t^{m+1}}{j!}\sum_{\mu=0}^{m-j}\left(-1\right)^{\mu}\binom{m+\mu}{\mu}\left[\sum_{\nu=\mu}^{m-j}\binom{m+\nu}{m+\mu}\right]t^{\mu}.$$

Therefore we obtain that

$$B_{m,j}(t) = \frac{(t-1)^j\ t^{m+1}}{j!}\sum_{\mu=0}^{m-j} b_{m,j}^{(\mu)} t^{\mu},$$

where

$$(7)\qquad b_{m,j}^{(\mu)} = (-1)^{\mu}\binom{m+\mu}{\mu}\binom{2m-j+1}{m+\mu+1},\ j=\overline{0,m},\ \mu=\overline{0,m-j}.$$

Using these expressions the second group of recurrence relations is obtained.

Proposition 5. The fundamental Hermite interpolating polynomials $A_{m,i}(t)$ and $B_{m,j}(t)$ satisfy

$$(8)\quad \begin{cases} A_{m,i}^{(m+1)}(0) = -\dfrac{(m+1)\ (2m-i+1)!}{i!(m+1-i)!}, \\ A_{m,i}^{(m+1)}(1) = (-1)^{m+1}\dfrac{(2m-i+1)!}{i!(m-i)!},\ i=\overline{0,m}\ , \end{cases}$$

and

$$(9)\quad \begin{cases} B_{m,j}^{(m+1)}(0) = (-1)^{j}\dfrac{(2m-j-1)!}{j!(m-j)!}, \\ B_{m,j}^{(m+1)}(1) = (-1)^{m-j}\dfrac{(m+1)(2m-j+1)!}{j!(m+1-j)!}\ ,\ j=\overline{0,m}\ . \end{cases}$$

Proof. Firstly, we have successively

$$A_{m,i}^{(m+1)}(1) = \frac{(-1)^{m+1}}{i!}\sum_{\nu=0}^{m-i}\binom{m+\nu}{\nu}\frac{d^{m+1}}{dt^{m+1}}\left[t^{i+\nu}(t-1)^{m+1}\right]_{t=1} =$$

$$= \frac{(-1)^{m+1}}{i!}\sum_{\nu=0}^{m-i}\binom{m+\nu}{\nu}(m+1)! = \frac{(-1)^{m+1}(m+1)!}{i!}\binom{2m-i+1}{m+1} =$$

$$= \frac{(-1)^{m+1}(2m-i+1)!}{i!(m-i)!}\ .$$

Taking into account that

$$B_{m,j}(t) = \frac{(t-1)^{j}\ t^{m+1}}{j!}\sum_{\mu=0}^{m-j} b_{m,j}^{(\mu)}t^{\mu} = \sum_{\mu=0}^{m-j} b_{m,j}^{(\mu)}(t-1)^{j}t^{m+1+\mu}\ ,$$

and using (7) it results

$$B_{m,j}^{(m+1)}(1) = \binom{m+1}{j}\sum_{\mu=0}^{m-j}(-1)^{\mu}\binom{m+\mu}{\mu}\binom{2m-j+1}{m+\mu+1}\frac{(m+\mu+1)!}{(\mu+j)!} =$$

$$= \frac{(m+1)(2m-j+1)!}{j!(m+1-j)!}\sum_{\mu=0}^{m-j}(-1)^{\mu}\binom{m+\mu}{\mu+j}\binom{m-j}{\mu}\ .$$

On the other hand, indentifying the coefficients of t^{m} in the two developments of $(1-t)^{m-j}\ (1-t)^{j-m-1} = (1-t)^{-1}$ we have in the right side that is 1 and in the left side is given by

$$(-1)^{m-j}\sum_{\mu=0}^{m-j}(-1)^{\mu}\binom{m-j}{\mu}\binom{m+\mu}{j+\mu}.$$

Hence it is obtained the second formula of (9).

The formulas for $B^{(m+1)}_{m,j}(0)$ and $A^{(m+1)}_{m,i}(0)$ are similarly calculated.

3. Existence and uniqueness of spline interpolation. Let $s_k = s_f(x_k)$, $k=\overline{0,n+1}$, be considered as unknown parameters, excepting $s_o = f(x_o)$, and $s_{n+1} = f(x_{n+1})$. On each subinterval $\left[x_k, x_{k+1}\right]$ there exists uniquely the Hermite interpolating polynomial of degree 2m+1 which matches s_k and s_{k+1}, and the first m derivatives of f at x_k and x_{k+1}. The spline interpolation function s_f on $\left[x_k, x_{k+1}\right]$ is considered to coincide with this polynomial. It remains to determine the parameters s_k, $k=\overline{1,n}$, from the continuity conditions

$$(10) \qquad s_f^{(m+1)}(x_k-0) = s_f^{(m+1)}(x_k+0), \quad k=\overline{1,n}.$$

Theorem 6. The considered Hermite-Birkhoff spline interpolation problem has a unique solution $s_f \in \mathcal{S}_{2m+1}(\Delta_n)$ if and only if

(i) m is an even positive integer and n is any positive integer, or

(ii) m is an odd positive integer and n is an even positive integer.

<u>Proof</u>. When $x \in \left[x_k, x_{k+1}\right]$ using lemma 3 it results that

$$s_f(x) = A_{m,o}(t)s_k + B_{m,O}(t)s_{k+1} + \sum_{i=1}^{m}\left[A_{m,i}(t)f_k^{(i)} + B_{m,i}(t)f_{k+1}^{(i)}\right]h^i.$$

Then we have

$$s_f^{(m+1)}(x_k-0) = \frac{1}{h^{m+1}}\Big\{A^{(m+1)}_{m,o}(1)s_{k-1} + B^{(m+1)}_{m,o}(1)s_k + \\ + \sum_{i=1}^{m} h^i\left[A^{(m+1)}_{m,i}(1)f^{(i)}_{k-1} + B^{(m+1)}_{m,i}(1)f^{(i)}_k\right]\Big\}$$

and

$$s_f^{(m+1)}(x_k+0) = \frac{1}{h^{m+1}}\Big\{A^{(m+1)}_{m,o}(0)s_k + B^{(m+1)}_{m,o}(0)s_{k+1} + \\ + \sum_{i=1}^{m} h^i\left[A^{(m+1)}_{m,i}(0)f^{(i)}_k + B^{(m+1)}_{m,i}(0)f^{(i)}_{k+1}\right]\Big\},$$

and based on the continuity conditions (10) the linear algebraic system follows

$$A_{m,o}^{(m+1)}(1)s_{k-1}+\left[B_{m,o}^{(m+1)}(1)-A_{m,o}^{(m+1)}(0)\right]s_k-B_{m,o}^{(m+1)}(0)s_{k+1}=$$

$$=\sum_{i=1}^{m} h^i\left\{-A_{m,i}^{(m+1)}(1)f_{k-1}^{(i)}+\left[A_{m,i}^{(m+1)}(0)-B_{m,i}^{(m+1)}(1)\right]+B_{m,i}^{(m+1)}(0)f_{k+1}^{(i)}\right\},$$

for $k=\overline{1,n}$, with $s_o=f_o$, and $s_{n+1}=f_{n+1}$.

An equivalent form of this system is obtained when proposition 5 is considered:

$$-s_{k-1}+\left[1+(-1)^m\right]s_k-(-1)^m s_{k+1}=$$

$$=\sum_{i=1}^{m}\frac{\binom{m}{i}}{\binom{2m+1}{i}}\frac{h^i}{i!}\left\{f_{k-1}^{(i)}-\frac{m+1}{m-i+1}\left[(-1)^m+(-1)^i\right]f_k^{(i)}+(-1)^{m-i}f_{k+1}^{(i)}\right\}=b_k,$$

for $k=\overline{1,n}$.

For even m the determinant of this linear system is different from zero and so it has a unique solution. Therefore, in this case $s_f\in\mathscr{S}_{2m+1}(\Delta_n)$ uniquely exists.

When m is an odd positive integer and n is even the solution of the linear algebraic system is given by

$$s_{k+1}=s_{k-1}+b_k\ ,\ k=1,3,\ldots,n-1,$$

and

$$s_{k-1}=s_{k+1}-b_k\ ,\ k=n,n-2,\ldots,2.$$

this finished the proof of theorem.

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