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GENERALIZED CONTRACTIONS IN UNIFORM SPACES

Vasile BERINDE

1. INTRODUCTION

Banach's fixed point principle in metric spaces was generalized in [9], [10] to uniform spaces as follow (see also [12]).

THEOREM 1. (N. Gheorghiu's theorem).

Let X be a Haussdorf sequentially complete uniform space with uniformity defined by a saturated family of pseudometrics $(d_i)_{i\in I}$, I being an index set. Let $f: X \to X$ a map with the properties

1) There exists $\alpha: I \to I$, $q: I \to \mathbb{R}_+$ such that

$$d_i(f(x),f(y)) \le q_i d_{\alpha(i)}(x,y) \ , \quad \forall\, i\in I \ , \quad \forall\, x,y\in X;$$

2) The series

$$\sum_{n=1}^{\infty} q_i q_{\alpha(i)} \ldots q_{\alpha^n(i)} (x,y)$$

is convergent for each $i\in I$ and each $x,y\in X$.

Then f has a unique fixed point.

Also, the same fixed point principle was extended in [2] to generalized contraction principle.

THEOREM 2.

Let (X,d) be a complete metric space and $f: X \to X$ a φ-contraction with φ a (c)-comparison function, i.e. a map which satisfies the following condition

$$d(f(x),f(y)) \le \varphi(d(x,y)) \ , \quad \forall x,y\in X. \tag{1}$$

Then f has a unique fixed point $x^* \in X$, and $f^n(x_0) \to x^*$ as $n \to \infty$ for each $x_0 \in X$ (here f^n stands for the n-th iterate of f).
Moreover

$$d(f^n(x_0), x^*) \le s(d(x_n, x_{n+1})) \ , \quad n \in \mathbf{N} \ , \tag{2}$$

where s(t) denote the sum of the comparison series

$$\sum_{k=0}^{\infty} \varphi^k(t) \ , \quad t \in \mathbb{R}_+ \ . \tag{3}$$

The purpose of this paper is to extend theorem 1 using the concepts involved in theorem 2.

2. COMPARISON FUNCTIONS

Referring to generalized φ-contractions we shall folow, both in terminology and notation, the monograph [12] and the papers [13], [2], [3].

DEFINITION 1. A function $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ is called (c)-<u>comparison function</u> if

(u) φ is monotone increasing;

(v) There exist two numbers $k_0, b, 0 \le b < 1$ and a convergent series with nonnegative real terms $\sum_{k=1}^{\infty} a_k$, such that

$$\varphi^{k+1}(t) \le b \cdot \varphi^k(t) + a_k \ , \quad for \ k \ge k_0 \ , \quad \forall t \in \mathbb{R}_+ \ . \tag{4}$$

REMARKS.

1) If φ ia a (c)-comparison function then (see [2]-[5]) the series (3) converges for all $t \in \mathbb{R}$. Moreover, the condition (4) is necessary and sufficient for the convergence of this series;

2) For every comparison function we have $\varphi(t) < t$, if $t > 0$, hence every φ-contraction is a continuous mapping;

3) For $\varphi(t) = at$, $0 \le a < 1$, from theorem 2 we obtain the well known Banach mapping principle;

4) There exist nonlinear comparison functions and also, discontinuous comparison functions, see [13], [3].

Thus, the study of generalized contractions, in metric or uniform spaces, is motivated by theoretical and applicative arguments.

To obtain the main result of this paper we need the following concept.

DEFINITION 2. We say that a familly $(\varphi_i)_{i\in I}$, $\varphi_i: \mathbb{R}_+ \to \mathbb{R}$, $i\in I$, is a (c)-comparison functions family with respect to $\alpha: I \to I$ if

(i) φ_i is monotone increasing, for each $i\in I$;

(ii) For each $i\in I$, there exist a convergent series with nonnegative real terms $\sum_{k=1}^{\infty} a_k^{(i)}$ and a number $b\in[0,1)$, such that for each function $t: I \to \mathbb{R}$, the following condition holds

$$(\varphi_i\circ\varphi_{\alpha(i)}\circ\ldots\circ\varphi_{\alpha^k(i)})(t_i) \le b\cdot(\varphi_i\circ\varphi_{\alpha(i)}\circ\ldots\circ\varphi_{\alpha^{k-1}(i)})(t_{\alpha(i)}) + a_k^{(i)}\ , \quad \text{for } k\ge N \text{ (fixed)} \quad . \tag{5}$$

LEMA 1. *If $(\varphi_i)_{i\in I}$ is a family of (c)-comparison functions with respect to $\alpha: I \to I$, then the series*

$$t_i + \sum_{k=1}^{\infty} (\varphi_i\circ\varphi_{\alpha(i)}\circ\ldots\circ\varphi_{\alpha^k(i)}(t_{\alpha(i)}) \ , \tag{6}$$

is convergent, for each $t: I \to \mathbb{R}$ and every $i\in I$.

PROOF. We apply theorem 2 [4].

REMARK. A family of (c)-comparison functions with respect to the identical map $\alpha=1$, is, in fact, a family of (c)-comparison functions in the usual sense.

3. φ- CONTRACTIONS IN UNIFORM SPACES

The main result of this paper is the following.

THEOREM 3.

Let X be a Haussdorf sequentially complete uniform space with uniformity defined by a saturated family of pseudometrics $(d_i)_{i\in I}$, and let $f: X \to X$ be a map with the property that there exist a function $\alpha: I \to I$ and a family $(\varphi_i)_{i\in I}$ of (c)-comparison functions with respect to α, such that

$$d_i(f(x),f(y)) \le \varphi_i(d_{\alpha(i)}(x,y)) \ , \quad \forall x,y\in X \ , \quad \forall i\in I \ . \tag{7}$$

Then f has a unique fixed point x^*, and $f^n(x_0) \to x^*$ as $n \to \infty$ for each $x_0 \in X$.
Moreover

$$d_i(x_n, x^*) \le s_i(d_i(x_n, x_{n+1})) \ , \quad i \in I \ , \tag{8}$$

where $s_i(t)$ denote the sum of the series (6).

PROOF.

Let $x_0 \in X$ and (x_n), $x_n = f^n(x_0)$, the sequence of successive approximations. We have

$$d_i(x_{n+2}, x_{n+1}) \le d_i(f(x_{n+1}), f(x_n)) \le \varphi_i(d_{\alpha(i)}(x_{n+1}, x_n)),$$

and

$$d_i(x_{n+3}, x_{n+2}) \le \varphi_i(d_{\alpha(i)}(x_{n+2}, x_{n+1})) \le (\varphi_i \circ \varphi_{\alpha(i)})(d_{\alpha(i)}(x_{n+1}, x_n)),$$

Then, by introduction,

$$d_i(x_{n+p}, x_{n+p-1}) \le (\varphi_i \circ \varphi_{\alpha(i)} \circ \ldots \circ \varphi_{\alpha^{p-2}(i)})(d_{\alpha^{p-1}(i)}(x_{n+1}, x_n)),$$

hence, for each $i \in I$ and each $n, p \in \mathbf{N}$ we have

$$d_i(x_{n+p}, x_n) \le d_i(x_{n+1}, x_n) + \sum_{k=1}^{p-1} (\varphi_i \circ \varphi_{\alpha i)} \circ \ldots \circ \varphi_{\alpha^{k-1}(i)})(d_{\alpha^k(i)}(x_{n+1}, x_n)). \tag{9}$$

Since $(\varphi_i)_{i \in I}$ is a familly of (c)-comparison function with respect to α it results that (x_n) is a Cauchy sequence. But X is a sequentially complete uniform space. Hence (x_n) is convergent.
By an other hand, the contraction condition implies, using Remark 2, the continuity of f. This means x^*, the limit of the sequence (x_n), is a fixed point of f.

To prove the unicity, assume there exists $y^* \in X$, $y^* \neq x^*$ an other fixed point of f. Then

$$d_i(x^*, y^*) = d_i(f(x^*), f(y^*)) \le \ldots \le (\varphi_i \circ \varphi_{\alpha(i)} \circ \ldots \varphi_{\alpha^n(i)})(d_{\alpha^{n+1}(i)}(x^*, y^*))$$

which yealds, on the basis of the convergence of (6),

$$(\varphi_i \circ \varphi_{\alpha(i)} \circ \ldots \circ \varphi_{\alpha^n(i)})(t_{\alpha^n(i)}) \quad \to 0 \text{ as } n \to \infty.$$

Thus we obtain

$$d_i(x^*,y^*) = 0\ ,\quad \textit{for each}\ \ i\in I\ . \tag{10}$$

But $(d_i)_{i\in I}$ is a saturated family of pseudometrics, hence, for each $x,y\in X$, $x\neq y$, there exists $i\in I$ such that

$$d_i(x,y) \neq 0,$$

which shows that (10) is false. This contradiction implies the unicity of x^*.

To obtain (8) it suffices to take $p \to \infty$ in (9). Thus, the proof is complete.

REMARK.

1) For $\varphi_i(t)=q_i t$, where $q\in\mathbb{R}_+$, $i\in I$, from theorem 3, we obtain a result which completes the Gheorghiu's fixed point theorem [9] by the estimation (8);

2) The same idea may by applied to other fixed point theorems in uniforms spaces [7], [8], [1], [11]. From these papers, only the work of Heikkila and Seikkala gives the estimation of the convergence, but in terms of the minimal solution of the equation

$$t-\varphi(t) = d(x_0,f(x_0)),$$

where $\varphi(t)=(\varphi_i(t))_{i\in I}$, in the particular case $\alpha=1_I$.

3) If we consider the following generalized metric (K-metric) $d\colon X\times X \to K$,

$$(x,y) \xmapsto{\ d\ } (i \mapsto d_i(x,y)),$$

where $K = \mathbb{R}_+^I$ is the cone of all functions $u\colon I \to \mathbb{R}_+$, then the estimation (8) may by written in the form (see[6])

$$d(x_n,x^*) \le s(d(x_n,x_{n+1})),$$

(here t denotes $(t_i)_{i\in I}$ and $s(t)$ denotes $(s_i(t_i))_{i\in I}$).

4) The fixed point theorems in uniform spaces have applications to probabilistic metric spaces, see [11] and also may be applied in order to obtain existence - uniqueness results for functional differential equations [1].

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UNIVERSITY OF BAIA MARE
DEPARTMENT OF MATHEMATICS
4800 BAIA MARE
ROMÂNIA