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NEW INEQUALITIES OBTAINED BY MEANS OF THE QUADRATURE FORMULAE

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Abstract. New inequalities are obtained by means of the quadrature formulae. The results of [3] are extended.

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In [1] we has studied two procedures of using the quadrature formulae in obtaing inequalities. In this paper we obtain new inequlities by these methods.

I). Let $w:[a,b] \to (0,\infty)$ be a weight function.

Preposition 1. For each polynomial $p_{2m}(x) \geq 0, x \in [a,b]$, degree 2m and with dominant coefficient equal 1, the inequality

(1)
$$\int_a^b w(x)\ p_{2m}(x)\,dx \geq \frac{1}{a_m^2}\int_a^b w(x)\ Q_m^2(x)\,dx$$

is valid, with equality only if

$$p_{2m}(x) = \frac{1}{a_m^2}\,Q_m^2(x)\ ,$$

where $Q_m(x)$ is the polynomial of degree m, with the dominant coefficient a_m, out of the system of orthogonal polynomials on the interval [a,b] refering to the weight w(x).

Proof. The validity of Preposition is obtained from the Gauus quadrature formula (see [2],[3])

$$\int_a^b w(x)\ f(x)dx = \sum_{i=1}^m A_i\ f(x_i) + R_{2m-1}(f)\ ,$$

in which the coefficients $A_i, i=\overline{1,m}$, are positive and the remainder is given by

$$R_{2m-1}(f) = \frac{1}{(2m)!}\,\frac{1}{a_m^2}\,f^{(2m)}(c)\int_a^b w(x)\ Q_m^2(x)dx\ ,\ c\in(a,b).$$

Remarks 1. For $w(x)=(1-x)^{\alpha}(1+X)^{\beta}, x\in(-1,1), \alpha>-1, \beta>-1$, from (1) it results the inequality given by F. Locher in [5].

Remarks 2. For $w(x)=x^{\alpha}e^{-x}, x\in(0,+\infty), \alpha>-1$, we obtain the Preposition 5 from [1].

Remarks 3. If $w(x)=e^{-x^2}, x\in(-\infty,+\infty)$, then for each polynomial $p_{2m}\geq 0, x\in(-\infty,+\infty)$, of degree 2m and with the dominant coefficient equal to 1, the inequality

$$\int_{-\infty}^{+\infty} e^{-x^2}\ p_{2m}(x)dx \geq \frac{m!\sqrt{\pi}}{2^m}\ ,$$

is valid,with equality only if

$$p_{2m}(x) = \frac{1}{2^{2m}} H_m^2(x) ,$$

where $H_m(x)$ is the Hermite polynomial.

II).The Gauss-Kronrod quadrature formula for the Legendre weight function,$w(x) = 1$,on $[-1,1]$,has the form

(2)

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n+1} A_i f(x_i) + \sum_{j=1}^{n} B_j f(a_j) + R_n(f) ,$$

where $a_j, j=\overline{1,n}$,are the zeros of the nth degree Legendre polynomial, $P_n(x)$,and the $x_i, A_i, i=\overline{1,n+1}, B_j, j=\overline{1,n}$,are chosen such (2) has maximum degree of exactness($3n+1$ for n even or $3n+2$ if n is odd).It is known that the x_i are simple,all contained in the interval $(-1,1)$ and they interlace with a_j that is

(3) $$x_{n+1}<a_n<x_n<\ldots<x_3<a_2<x_2<a_1<x_1$$

(see [7]-[9]).Moreover,all coefficients of (2) are positive(the positiving of A_i is equivalent to the interlacing property (3);see [5]).

Let us $f\in C^{(3n+2)}[-1,1]$ and n even,then

$$R_n(f) = \frac{(n!)^2}{2^n(3n+2)!(2n)!} \int_{-1}^{1} P_n(x)\, w_{n+1}^2(x) \quad f^{(3n+2)}(C_x)\, dx \, ,C_x\in(-1,1) ,$$

where

$$w_{n+1}(x) = \prod_{i=1}^{n+1}(x-x_i)$$

and it satisfies the following orthogonality relation

$$\int_{-1}^{1} p_n(x) w_{n+1}(x) x^k dx = 0 \, , \; k=\overline{0,n}.$$

When n is odd,if we assume $f\in C^{(3n+2)}[-1,1]$,then

$$R_n(f) = \frac{n!}{2^n(3n+3)!(2n)!} \int_{-1}^{1} P_n(x)\, w_{n+1}^2(x) \quad f^{(3n+3)}(C_x)\, dx ,$$

$C_x\in(-1,1)$ (see [7]).Now we obtain:

<u>Preposition 2.</u>If n is even,then for each polinomial $p_{3n+2}(x)\geq 0$, $x\in \in[-1,1]$,of degree $3n+2$ and with the coefficient equal 1,the inequality

$$\int_{-1}^{1} p_{3n+2}(x)dx \geq \frac{(n!)^2}{2^n(2n)!} \int_{-1}^{1} P_n(x) w_{n+1}^2(x)dx \text{ is true.}$$

If n is odd,then for each polynomial $p(x)\geq 0, x\in[-1,1]$,of degree $3n+3$ and with the dominant coefficient equal 1,the inequality

$$\int_{-1}^{1} p_{3n+3}(x)dx \geq \frac{(n!)^2}{2^n(2n)!} \int_{-1}^{1} P_n(x)\, w_{n+1}^2(x)dx \text{ is valid.}$$

III).Let's consider the Fuler's quadrature formula(see [2],[4])

(4)
$$\int_a^b F(x)dx = \frac{b-a}{2}[f(a)+f(b)]+$$

$$+\sum_{i=1}^{n-1}\frac{(b-a)^{2i}}{(2i)!}B_{2i}\left[f^{(2i-1)}(a)-f^{(2i-1)}(b)\right]+R(f),$$

with

(5)
$$R(f) = -\frac{(b-a)^{2n-1}B_{2n}}{(2n)!}f^{(2n)}(c),\ c\in(a,b),$$

where $B_{2j}, j=\overline{1,n}$,are the Bernoulli numbers.If $f\in C^{(2n)}[a,b]$,with$f^{(2n)}(x)\geq 0$ for any $x\in[a,b]$,and $B_{2n}>0$,then from (4) and (5) we obtained the inequality

(6)
$$\int_a^b f(x)dx < \frac{b-a}{2}[f(a)+f(b)]+$$

$$+\sum_{i=1}^{n-1}\frac{(b-a)^{2i}}{(2i)!}B_{2i}\left[f^{(2i-1)}(a)-f^{(2i-1)}(b)\right].$$

If $B_{2n}<0$,then we have the inequality

(7)
$$\int_a^b f(x)dx \geq \frac{b-a}{2}[f(a)+f(b)]+$$

$$+\sum_{i=1}^{n}\frac{(b-a)^{2i}}{(2i)!}B_{2i}\left[f^{(2i-1)}(a)-f^{(2i-1)}(b)\right].$$

For $f^{(2n)}(n)\leq 0$ on $[a,b]$,the inequality (6) and (7) reverse the order.

The inequalities (6) and (7) generalize the results from [1].

If in (6) we insert $f(x)=1/x, x\in[a,b], 0<a<b$,then we find the inequality

$$\ln\frac{b}{a} < \frac{b^2-a^2}{2ab}.$$

From here,for $a=1, b=1+x, x>0$,it results

$$\ln(1+n) < \frac{x(x+2)}{2(x+1)}.$$

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