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ON R-n-MODULES

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Summary: This paper gives some elementary results on n-modules and on the category of n-modules; two constructions of the quotient n-module are given. In R-n-modules, non-uniqueness of the representation of an element relative to a basis is remarked.

1. **Introduction.** N. Celakoski [1] introduced the notion of R-n-module and studied some properties of projective and injective R-n-modules.

The aim of this paper is to give some results on R-n-modules, analogous to those concerning R-modules, that would allow further systematic research.

2. Let R be an associative ring with neutral element $(1\neq 0)$, $(A,\circ)$ an abelian n-group with only one neutral element, denoted O. A is said to be an R-n-module if there is defined a mapping φ: $R\times A \to A$, $(r,a)\to ra$ such that:

$$1a = a,\ (rs)a = r(sa),\ r(a_1,\ldots,a_n)_\circ = (ra_1,\ldots,ra_n)_\circ,$$

$$\left(\sum_{i=1}^{n} r_i\right)a = (r_1a,\ldots,r_na)_\circ,\ \text{ for every } r,\ r_i\in R,\ a,\ a_i\in A,\ i=1,\ldots,n.$$

It is easy to see that $0\cdot a=0$, and $r\cdot 0=0$, for every $a\in A$, $r\in R$; we also have $(-1)\cdot a=(a,a,\ldots,a,\bar{a},0,0)_\circ$, and

$$[\underbrace{(-1)+(-1)+\ldots+(-1)}_{n-2}]\cdot a = \bar{a}\ .$$

A subset $B \subseteq A$ is said to be an n-submodule if operation from A induce operations in B and B with these operations is an R-n-module.

Let X be a subset of A, $X \subseteq A$. A finite sum

$x = (\ldots((r_1 x_1, \ldots, r_n x_n)_\circ r_{n+1} x_{n+1}, \ldots, r_{2n-1} x_{2n-1})_\circ \ldots r_s x_s)_\circ$,

where $s \equiv 1 \pmod{n-1}$, $r_k \in R$, $x_k \in X$, $k = 1, 2, \ldots, s$, is called a linear combination of elements from the subset X.

The n-submodule generated by X is defined as

$\langle X \rangle = \bigcap \{B \in \mathcal{S}_{Rn}(A) \mid X \subseteq B\}$ (where $\mathcal{S}_{Rn}(A)$ denotes the set of n-submodules of A).

If $X \subseteq A$, $X \neq \emptyset$ then

$$\langle X \rangle = \{(\ldots(r_1 x_1, \ldots, r_n x_n)_\circ, \ldots, r_s x_s)_\circ \mid r_k \in R,\ x_k \in X,\ k = 1, \ldots, s;\ s \equiv 1 \pmod{n-1}\}$$

$$\langle\{x\}\rangle = \langle x \rangle = \{rx \mid r \in R\}$$

Now, one can easy prove that:

Proposition 1 a) *The lattice $\mathcal{S}_{Rn}(A)$ of the n-submodules of A is a complete lattice.*

b) *$\mathcal{S}_{Rn}(A)$ is a complete sublattice of $\mathcal{S}_n(A)$ (which is the lattice of the n-subgroups of A).*

An equivalence ρ on A is said to be a congruence on the R-n-module A if:

$$a_1 \rho b_1, a_2 \rho b_2, \ldots, a_n \rho b_n \Rightarrow (a_1, a_2, \ldots, a_n)_\circ\, \rho\, (b_1, b_2, \ldots, b_n)_\circ$$

$$a \rho b,\quad r \in R \Rightarrow (ra)\, \rho\, (rb)$$

We shall denote the set of all congruences on the R-n-module A by $\mathcal{C}_{Rn}(A)$. $(\mathcal{C}_{Rn}(A), \subseteq)$ is a complete lattice.

Theorem 1. *The mapping $f: \mathcal{S}_{Rn}(A) \to \mathcal{C}_{Rn}(A)$, $f(B) = \rho_B$ (where ρ_B is defined as follows: $a_1 \rho_B a_2 \Leftrightarrow (a_1, a_2, \ldots, a_2, \overline{a_2}, 0)_\circ \in B$) is an isomorphism of lattices and $f^{-1}(\rho) = \rho\langle 0 \rangle = \{a \in A \mid 0 \rho a\}$.*

Proof a) We prove that ρ_B is a congruence on A. We have $(a_1, a_1, \ldots, a_1, \overline{a_1}, 0)_\circ = 0 \in B$; this shows that ρ_B is reflexive; $a_1 \rho_B a_2$

implies that $(a_1,a_2,\dots,a_2,\overline{a_2},0)_\circ \in B$ and, as B is an n-submodule,

$(-1)\cdot(a_1,a_2,\dots,a_2,\overline{a_2},0)_\circ \in B$; but

$(-1)\cdot(a_1,a_2,\dots,a_2,\overline{a_2},0)_\circ = ((a_1,a_2,\dots,a_2,\overline{a_2},0)_\circ,\dots,(a_1,a_2,\dots$

$\dots,a_2,\overline{a_2},0)_\circ,(\overline{a_1},\overline{a_2},\dots,\overline{a_2},\overline{a_2},0)_\circ,0,0)_\circ = (a_2,a_1,\dots,a_1,\overline{a_1},0)_\circ$

which proves that $a_2\rho_B a_1$, i.e. ρ_B is symmetrical.

$a_1\rho_B a_2$, $a_2\rho_B a_3$ implies that $(a_1,a_2,\dots,a_2,\overline{a_2},,0)_\circ \in B$ and

$(a_2,a_3,\dots,a_3,\overline{a_3},0)_\circ \in B$; then

$((a_1,a_2,\dots,a_2,\overline{a_2},0)_\circ,(a_2,a_3,\dots,a_3,\overline{a_3},0)_\circ, 0,\dots,0)_\circ \in B$, which

means $(a_1,a_3,a_3,\dots,a_3,\overline{a_3},0)_\circ \in B$, i.e. ρ_B is transitive.

If $a_1\rho_B b_1$, $a_2\rho_B b_2,\dots,a_n\rho_B b_n$ then $(a_1,b_1,\dots,b_1,\overline{b_1},0)_\circ \in B$,

$\dots,(a_n,b_n,\dots,b_n,\overline{b_n},0)_\circ \in B$ and also

$((a_1,b_1,\dots,b_1,\overline{b_1},0)_\circ,\dots,(a_n,b_n,\dots,b_n,\overline{b_n},0)_\circ)_\circ \in B$, which means

$((a_1,a_2,\dots,a_n)_\circ,(b_1,\dots,b_n)_\circ,\dots,(b_1,\dots b_n)_\circ,(\overline{b_1,\dots,b_n})_\circ,0)_\circ \in B$,

i.e. $(a_1,a_2,\dots,a_n)_\circ\,\rho_B(b_1,b_2,\dots,b_n)_\circ$.

If $a\rho_B b$ and $r\in R$ then

$(ra,rb,\dots,rb,\overline{rb},0)_\circ = (ra,rb,\dots,rb, r\overline{b},0)_\circ = r(a,b,\dots,b,\overline{b},0)_\circ \in B$,

which proves that $(ra)\rho_B(rb)$.

We now know that ρ_B is a congruence on A.

b) We prove that f is an isomorphism of lattices.

Let $B_1,B_2\in\mathscr{S}_{Rn}(A)$, $B_1\neq B_2$. This means that $\exists x\in B_1\setminus B_2$;

$x=(x,0,0,\dots,0)_\circ \in B_1 \Leftrightarrow x\rho_{B_1}0$, and $(x,0,0,\dots,0)_\circ \notin B_2 \Leftrightarrow x\not\rho_{B_2}0$.

This proves that $\rho_{B_1}\neq\rho_{B_2}$, i.e $f(B_1)\neq f(B_2)$, and f is injective.

Let ρ be a congruence on A, $\rho\in\mathscr{C}_{Rn}(A)$, and $B=\rho\langle 0\rangle=\{a\in A\mid a\rho 0\}$. It is easy to prove that $B\in\mathscr{S}_{Rn}(A)$ and that $\rho=\rho_B$; therefore f is surjective.

If $B_1\subseteq B_2$ and $a\rho_{B_1}b$, then $(a,b,b,\dots,\overline{b},0)\in B_1\subseteq B_2$;

$(a,b,\dots,b,\overline{b},0)_\circ \in B_2 \Leftrightarrow a\rho_{B_2}b$. We proved that $\rho_{B_1}\subseteq\rho_{B_2}$ i.e.

$f(B_1)\subseteq f(B_2)$, which completes the demonstration.

This isomorphism allows us to identify $\rho\in\mathscr{C}_{Rn}(A)$ with $B=\rho\langle 0\rangle\in\mathscr{S}_{Rn}(A)$, and to define $A/B=\{a+B|a\in A\}$, where $a+B=\{(a,b,0,\ldots,0)_\circ\ |b\in B\}$. A/B is an R-n-module with the following operations:

$$((a_1+B),\ (a_2+B),\ldots,(a_n+B))_\circ = (a_1,a_2,\ldots,a_n)_\circ+B$$

$$r(a+B) = ra+B$$

and is said to be the <u>quotient R-n-module</u>.

Theorem 2 *The mapping* $f':\mathscr{S}_{Rn}(A)\to\mathscr{C}_{Rn}(A)$, $f'(B)=\rho'_B$ *(where* ρ'_B *is defined as follows* $a\rho'_B c \Leftrightarrow \exists\ b_1,\ldots,b_{n-1}\in B$ *such that* $c=(a,b_1,\ldots,b_{n-1})_\circ$*) is an isomorphism of lattices and* $f'^{-1}(\rho) = \rho\langle 0\rangle = \{a\in A|a\rho 0\}$.

Proof a) We prove that ρ'_B is a congruence on A. We have $a=(a,0,\ldots,0)_\circ$ and $0\in B$; this shows that ρ'_B is reflexive.

$a_1\rho'_B a_2$ implies that $\exists\ b_1,\ldots,b_{n-1}\in B$ such that $a_2=(a_1,b_1,\ldots,b_{n-1})_\circ$; solving the equation we obtain $a_1=(a_2,\underbrace{b_{n-1},\ldots,b_{n-1}}_{n-3},\overline{b_{n-1}},\ldots,b_1,\ldots,b_1,\overline{b_1})_\circ$, and making the necessary associations we obtain that $a_1=(a_2,b'_1,\ldots,b'_{n-1})_\circ$, i.e. $a_2\rho'_B a_1$, which proves that ρ'_B is symmetrical.

$a_1\rho'_B a_2,\ a_2\rho'_B a_3$ implies that $\exists\ b_1,\ldots,b_{n-1},\ c_1,\ldots,c_{n-1}\in B$ such that $a_1=(a_2,b_1,\ldots,b_{n-1})_\circ$, $a_2=(a_3,c_1,\ldots,c_{n-1})_\circ$; by replacing a_2 in the first relation we obtain $a_1=((a_3,c_1,\ldots,c_{n-1})_\circ,\ b_1,\ldots,b_{n-1})_\circ=$ $=(a_3,(c_1,\ldots,c_{n-1},b_1)_\circ,\ b_2,\ldots,b_{n-1})_\circ=(a_3,b'_1,b_2,\ldots,b_{n-1})_\circ$, where $b'_1\in B$, i.e. ρ'_B is transitive. If $a_1\rho'_B c_1,\ a_2\rho'_B c_2,\ldots,a_n\rho'_B c_n$ then $\exists b^1_1,\ldots,b^1_{n-1},\ b^2_1,\ldots,b^2_{n-1},\ldots,b^n_1,\ldots,b^n_{n-1}\in B$ such that

$a_1=(c_1,b_1^1,\dots,b_{n-1}^1)_\circ$, $a_2=(c_2,b_1^2,\dots,b_{n-1}^2)_\circ,\dots,a_n=(c_n,b_1^n,\dots,b_{n-1}^n)_\circ$;

making the operation we obtain $(a_1,a_2,\dots,a_n)_\circ=$

$$=((c_1,c_2,\dots,c_n)_\circ,\underbrace{b_1^1,\dots,b_{n-1}^1,b_1^2,\dots,b_{n-1}^2,\dots,b_1^n,\dots,b_{n-1}^n}_{n(n-1)\ terms})_\circ=$$

$$=((c_1,c_2,\dots,c_n),b_1,b_2,\dots,b_{n-1})_\circ, i.e\ (a_1,a_2,\dots,a_n)_\circ\rho'_B(c_1,c_2,\dots,c_n)_\circ .$$

If $a\rho'_B c$ and $r\in R$, then $\exists b_1,\dots,b_{n-1}\in B$ such that $a=(c,b_1,\dots,b_{n-1})_\circ$

$ra=r(c,b_1,\dots,b_{n-1})_\circ=(rc,rb_1,\dots,rb_{n-1})_\circ$ and, as $rb_k\in B$, $k=1,\dots,n-1$

we proved that $(ra)\rho'_B(rc)$.

We now know that ρ'_B is a congruence on A.

b) We prove that f' is an isomorphism of lattices.

Let $B_1,B_2\in\mathscr{S}_{Rn}(A)$, $B_1\neq B_2$. This means that $\exists b\in B_1\setminus B_2$. Let $c=(a,b,0,\dots,0)_\circ$; then $a\rho'_{B_1}c$. We shall prove that $\nexists b_1,\dots,b_{n-1}\in B_2$ such that $a=(c,b_1,\dots,b_{n-1})_\circ$, i.e. $a\not\rho'_{B_2}c$. If $\exists b_1,\dots,b_{n-1}\in B_2$ such that $a=(c,b_1,\dots,b_{n-1})$, then $c=((c,b_1,\dots,b_{n-1})_\circ, b,0,\dots,0)_\circ$ and therefore

$$b=(\underbrace{b_{n-1},\dots,b_{n-1}}_{n-3},\overline{b_{n-1}},\dots,\underbrace{b_1,\dots,b_1}_{n-3},\overline{b_1},\underbrace{c,\dots,c}_{n-3},\overline{c},c,\underbrace{0,\dots,0}_{(n-2)^2})_\circ=$$

$$=(b_{n-1},\dots,b_{n-1},\overline{b_{n-1}},\dots,b_1,\dots,b_1,\overline{b_1},0)_\circ\in B_2 ,$$ contradiction. This proves that $\rho'_{B_1}\neq\rho'_{B_2}$ i.e. $f'(B_1)\neq f'(B_2)$ and f is injective.

Let ρ be a congruence on A, $\rho\in\mathscr{C}_{Rn}(A)$ and $B=\rho\langle 0\rangle=\{a\in A/a\rho 0\}$. It is easy to prove that $B\in\mathscr{S}_{Rn}(A)$ and $\rho=\rho'_B$, therefore f' is surjective.

Now if $B_1\subseteq B_2$ and $a\rho'_{B_1}c$ then $\exists b_1,\dots,b_{n-1}\in B_1\subseteq B_2$ such that $a=(c,b_1,\dots,b_{n-1})_\circ$ i.e. $a\rho'_{B_2}c$ so $f'(B_1)\subseteq f'(B_2)$ . .

We have proved that f' is an isomorphism of lattics and $f'^{-1}(\rho)=\rho\langle 0\rangle$.

The isomorphism defined in theorem 2 allows us to identify $\rho\in\mathscr{C}_{Rn}(A)$ with $B=\rho\langle 0\rangle\in\mathscr{S}_{Rn}(A)$; then $A/B=\{a+(n-1)B|a\in A\}$, where

$a+(n-1)B=\{(a,b_1,\ldots,b_{n-1})_\circ | b_i \in B,\ i=1,\ldots,n-1\}$. A/B is an R-n-module with the following operations:

$$((a_1+(n-1)B),\ (a_2+(n-1)B),\ldots,(a_n+(n-1)B))_\circ = (a_1,\ldots,a_n)_\circ+(n-1)B$$
$$r(a+(n-1)B) = ra+(n-1)B$$

This second construction of the quotient n-module brings us to the notion introduced by Celakoski in [1].

The isomorphism theorems for R-n-modules can be obtained from the isomorphism theorems for universal algebras by identifying the congruences of an n-module with its n-submodules.

3. We shall denote the category of R-n-modules by Mod_{Rn}. In this category {0} is a zero object; therefore Mod_{Rn} is a category with zero morphisms ($\forall A,B\in Mod_{Rn}$, $O_{AB}:A \to B$, $O_{AB}(a)=O_B$, $\forall a\in A$).

Theorem 3 *The category Mod_{Rn} is with kernels and cokernels.*

Proof Let $f: A \to B$ be a homomorphism in Mod_{Rn}, $N=\{a\in A | f(a)=0\}$, $i:N \to A$, $i(a)=a$, $\forall a\in N$; $p: B \to B/f(A)$, $p(b)= b+f(A)$. Then ker $f=[N,i]$ and Coker $f=[p, B/f(A)]$.

Proposition 2 (see [1]). *Let A,B be R-n-modules and $f: A \to B$ a homomorphism. The following sentences are equivalent:*

a) f is an injective homomorphism

b) f is a monomorphism in Mod_{Rn}

c) For any R-n-module A' and any homomorphism $\alpha: A' \to A$, $f\circ\alpha=0$ implies $\alpha=0$

d) Ker f={0}

Proposition 3 (see [1]). *Let A,B be R-n-modules and $f: A \to B$ a homomorphism. The following sentences are equivalent:*

a) f is a surjective homomorphism

b) f is an epimorphism in Mod_{Rn}

c) For any R-n-module B' and any homomorphism $\beta: B \to B'$, $\beta\circ f=0$ implies $\beta=0$

d) Coker f={0}

Corollary *Mod_{Rn} is a perfect category.*

Theorem 4 *Mod_{Rn} is an exact category.*

Proof a) We prove that Mod_{Rn} is a normal and conormal category. Let A,B be R-n-modules and F: A → B a monomorphism in Mod_{Rn}. Then [A,f]=ker p, where p: B → B/f(A) is the homomorphism defined by p(b)=b+f(A).

If f: A → B is an epimorphism in Mod_{Rn}, then [f,B]=Coker i, where i: ker f → A is the homomorphism defined by i(a)=a.

b) Mod_{Rn} is a category with kernels and cokernels (see theorem 3).

c) Finally we prove that every morphism from Mod_{Rn} can be decomposed in a product of a monomorphism and an epimorphism.

Let f: A → B be a morphism in Mod_{Rn}; then g: A → f(A), g(a)=f(a) is an epimorphism, and i: f(A) → B, i(b)=b is a monomorphism and $f=i\circ g$.

4. **Remark.** If we define linear independence in R-n-modules in the same way as in usual R-modules, this brings us to the analogous notions of basis and free R-n-modules. Unfortunately, further results cannot be obtained because of the non-uniqueness of the coordinates of an element relative to the basis of the R-n-module. Though the following result holds:

Proposition 4. *A is a free R-n-module if and only if $\overline{A}$ is a free R-module (where the binary operation in $\overline{A} = (A,\hat{+})$ is defined as in [1]: $a\hat{+}b = (a,b,0,\ldots,0)_\circ$.)*

Proof. Let $B\subseteq A$ be a basis of $\overline{A}$. Then $\forall x\in A$ $\exists b_1,\ldots,b_k\in B$, $r_1,\ldots,r_k\in \mathbf{R}$ such that $x = r_1b_1 \hat{+} r_2b_2\hat{+}\ldots\hat{+}r_kb_k$; therefore

$$x = (\ldots((r_1b_1,r_2b_2,0,\ldots,0)_\circ,r_3,b_3,0,\ldots,0)_\circ\ldots)_\circ,r_kb_k,0,\ldots,0)_\circ =$$
$$= (r_1b_1,\ r_2,b_2,\ldots,r_kb_k,0,\ldots,0)_\circ$$

if $k\le n$, or $x = (\ldots(r_1b_1,\ r_2b_2,\ldots,r_nb_n)_\circ,r_{n+1}b_{n+1},\ldots,r_kb_k,0,\ldots,0)_\circ$ if $k>n$.

Anyway, x is a linear combination of elements from B.

If $b_1,\ldots,b_s\in B$, $s\equiv 1 (\mathrm{mod}\ n-1)$ and $(\ldots(r_1b_1,\ldots,r_nb_n)_\circ,\ldots,r_sb_s)_\circ = 0$

then $r_1 b_1 \hat{+} \ldots \hat{+} r_s b_s = 0$ which implies (as B is a free set in $\overrightarrow{A}$) that $r_1 = \ldots = r_s = 0$. This proves that B is also free in A.

The converse assertion is proved in an analogous way.

R E F E R E N C E S

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