Buletinul Ştiinţific al Universităţii din Baia Mare
Seria B, Matematică-Informatică, vol.IX(1993), 81-90

ON THE REDUCTION AND THE EXTENSION OF (m,n)-RINGS

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The concept of (m,n)-rings was introduced in 1965, by Cupona [4] and in a speciale case (n=2) by Boccioni [1]. Further, (m,n)-rings were examined by Crombez [2],[4] Purdea [11], Dudek [6], Leeson-Butson [7], in which some familiar results for ordinary rings (m=n=2) were generalized. Also, Boccioni [1] respectively Crombez [3] and Leeso-Butson [7] proved that a generalization of the Post coset theorem [9,p 218] could be obtained for (m,2)-rings, respectively for (m,n)-rings.

In this paper we define some extensions and reduces of (m,n)-rings and the connection between them through the change of the n-semigroup operation of the (m,n)-ring.

1. Definitions, notations and preliminary results.

An algebra $(R,+,\circ)$ is an *(m,n)-ring*, $m,n\geq 2$, if:

1) $(R,+)$ is a commutative m-group;

2) $(R,\circ)$ is an n-semigroup, and,

3) the following distributive laws hold for all choices of $a_1,\ldots,a_n,b_1,\ldots,b_m\in R$ and for all choices of $i\in\{1,2,\ldots,n\}$:

$$(a_1,\ldots,a_{i-1},(b_1+\ldots+b_m),a_{i+1},\ldots,a_n)_\circ=(a_1,\ldots,a_{i-1},b_1,a_{i+1},\ldots,a_n)_\circ+\ldots$$
$$\ldots+(a_1,\ldots,a_{i-1},b_m,a_{i+1},\ldots,a_n)_\circ\ . \quad (1)$$

Clearly, an ordinary ring is a (2,2)-ring.

In keeping the practice adopted for polyadic groups and semigroups, breafly notational convenience will be used, as follows:

$$(x_1+\ldots+x_j+\underbrace{x+\ldots+x}_{k\text{-times}}+x_{j+k+1}+\ldots+x_m) = \sum_{i=1}^{j} x_i+kx+\sum_{i=j+k+1}^{m} x_i$$

and $$(x_1,\ldots,x_j,\underbrace{x,\ldots,x}_{k\text{-times}},x_{j+k+1},\ldots,x_n)_\circ = (x_1^j,\overset{(k)}{x},x_{j+k+1}^n)_\circ .$$

Therefore the distributive laws can be written

(1') $(a_1^{i-1},\sum_{j=1}^{m} b_j,a_{i+1}^n)_\circ = \sum_{j=1}^{m} (a_1^{i-1},b_j,a_{i+1}^n)_\circ$.

An element $b\in R$ is an *additive idempotent* if $mb=b$ and b is a *multiplicative idempotent* if $\overset{(n)}{(b)}_0=b$ or, to power in the Dörnte's sense, $b^{[1]}=b$. If both of these conditions are satisfied, b will be called an *idempotent* of R. The element $\bar{b}$ will denote the *additive querelement* of b, so that b is the solution of the ecuation $(n-1)b+x=b$. It is easily seen that in an (m,n)-ring we have

(2) $\overline{b_1+\ldots+b_m} = \overline{b_1}+\ldots+\overline{b_m}$ and $(b_1,\ldots,\overline{b_i},\ldots,b_n)_\circ = (\overline{b_1,\ldots,b_n})_\circ$ for $i=1,\ldots,n$ and for all elements of R.

If the multiplicative querelement of $b\in R$ exists, then we will denote it by $\underline{b}$.

An element $0\in R$ is called a *zero* of R if $(x_1^{i-1},0,x_{i+1}^n)_\circ = 0$ for all $x_1,\ldots,x_n\in R$ and for all choices of $i\in\{1,2,\ldots,n\}$. A zero, if there is, clearly is an idempotent of R. An (m,n)-ring may have at most one zero. If R is a (2,n)-ring, then R has a zero element [7]. In this paper R^* will denote the set of non-zero elements in the (m,n)-ring R.

If $(u_1,\ldots,u_{n-1})$ is a right unit in the n-semigroup $(R,\circ)$, that is $(xu_1^{n-1})_\circ = x,\ \forall x\in\mathbf{R}$, then $(R,+,\circ)$ is called (m,n)-ring with right unit.

An (m,n)-ring $(R,+,\circ)$, is *cancellative* if the equation $(b_1^{i-1}a_i b_{i+1}^n)_\circ = (b_1^{i-1}c_i b_{i+1}^n)_\circ$ implies $a_i=c_i$ for each choice of $b_1,\ldots,b_n \in R^*$ and for each $i=1,2,\ldots,n$.
A commutative cancellative (m,n) ring is called an (m,n) *integral domain.*

An (m,n)-ring $(R,+,\circ)$ is an *(m,n)-division ring* if $(R^*,\circ)$ is an n-group. If $(R^*,\circ)$ is a commutative n-group, then $(R,+,\circ)$ is an *(m,n)-field.*

An element $b\in R$ is a *central element* in the (m,n)-ring R if $(bx_1^{n-1})_\circ = (x_1 b x_2^{n-1})_\circ = \ldots = (x_1^{n-1}b)_\circ$ for all $x_1,\ldots,x_{n-1}\in R$.

Leeson-Butson [7] proved that a finite (2,n)-division ring $(R,+,\circ)$ is an (2,n)-field if and only if there is a non-zero central element b in R and that a finite (m,n)-division ring R with a zero element is an (m,n)-field if only if R contains a non-zero central element.

We recall that if (R,φ) is a k-semigroup and $n=(k-1)s+1$; $s\in\mathbf{N}^*$, we define an n-ary operation, on R, called "long product" denoted by $\varphi_{(s)}$ or $\varphi_{(\cdot)}$ unless there is the possibility of confusion, as follows:

$$\varphi_{(s)}(x_1^n) = \varphi(\varphi(\ldots\varphi(\varphi(x_1^k), x_{k+1}^{2k-1}),\ldots), x_{n-k+1}^n)$$

It is obvious that an (m,n)-ring $(R,+,\circ)$ can be formed as an extension of the (p,k)-ring (R,ψ,φ), when $m=(p-1)t+1$; $n=(k-1)s+1$; $t,s\in\mathbf{N}^*$, by defining "+" as long product $\psi_{(t)}$ and "$\circ$" as $\varphi_{(s)}$. Clearly that every (m,n) ring formed as an extension of an ordinary ring has zero element. Leeson and Butson proved that an (m,n)-ring R with zero, extends a (2,n)-ring and as a corollary, an (m,2)-finite division ring with a zero is an (m,2)-field.

In sequel the extension of an (m,k)-ring R relatively to fixed elements and to an endomorphism of R is defined and also the reduce of some order of (m,k)-ring relatively to fixed elements is studied. These results allow constructions of new (m,n)-rings.

2. Reductions and extensions of (m,n)-rings.

Definition 1. Let $(R,+,\circ)$ be an (m,n)-ring, $k\in\mathbf{N}$; $k\geq 2$ so that $n-1=s(k-1)$, and $u_1,\ldots,u_{s-1}\in R$ fixed elements. The algebra $(R,+,*)$ where the operation $*: R^k \to R$ is defined by

$$(3)\qquad (x_1^k)_* = (x_1, u_1^{s-1}, x_2, u_1^{s-1}, \ldots, u_1^{s-1}, x_k)_\circ$$

is called *the reduce* of order (m,k) relatively to $(u_1,\ldots,u_{s-1})$ of R and is denoted by $\mathrm{red}\ {}^{m,k}_{u_1^{s-1}}(R,+,\circ)$.

It is easily seen that the following properties hold:

Proposition 1. *If $R,+,\circ)$ is an (m,n)-ring, $n-1=s(k-1)$; $s\in\mathbf{N}^*$, then for all $u_1,\ldots,u_{s-1}\in R$, not necessary distinct, red ${}^{m,k}_{u_1^{s-1}}(R,+,\circ)$ is an (m,k)-ring. If $(R,+,\circ)$ is an (m,n)-division ring then the reduce is an (m,k)-division ring isomorphic with the (m,k)-reduce relatively to aa...a, $\forall a\in R^*$, denoted by red ${}^{(m,k)}_{a}(R,+,\circ)$.*

Definition 2. Let $(R,+,\varphi)$ be an (m,k)-ring $n=s(k-1)+1$; $s\in\mathbf{N}^*$; let $c_1,\ldots,c_{k-1}$ be fixed elements of R, $\alpha\in\mathrm{End}\ (R,+,\varphi)$ and $\circ: R^n \to R$ the operation defined by

$$(4)\qquad (x_1^n)_\circ = \varphi_{(s+1)}(x_1, \alpha(x_2), \ldots, \alpha^{n-1}(x_n), c_1^{k-1}) \ .$$

The algebra $(R,+,\circ)$ is called *the (m,n)-ary extension* of the (m,k)-ring R relatively to the endomorphism α and to the elements $c_1,\ldots,c_{k-1}\in R$.

It is denoted by ***ext*** ${}^{m,n}_{\alpha,c_1^{k-1}}(R,+,\varphi)$.

Proposition 2. *If $(R,+,\varphi)$ is an (m,k)-ring $\alpha\in\mathrm{End}\ (R,+,\varphi)$; $c_1,\ldots,c_{k-1}\in A$; $n=s(k-1)+1$; $s\in\mathbf{N}^*$, so that the relation*

$$(5)\qquad \varphi(\alpha^n(x), \alpha(c_1), \ldots, \alpha(c_{k-1})), = \varphi(c_1^{k-1}, \alpha(x)),\ \forall x\in\mathbb{R}$$

holds, then the $\mathop{ext}\limits^{(m,n)}_{\alpha, c_1^{k-1}}(R,+,\varphi)$ *is an (m,n)-ring.*

Proof. Because $\alpha\in End(R,\varphi)$ by [8] the n-ary extension $(R,\circ)$ of the k-semigroup (R,φ) relatively to α and $c_1,\ldots,c_{k-1}$ is an n-semigroup.

But, for all $a_1,\ldots,a_n,b_1,\ldots,b_m\in\mathcal{R}$, and for all choices of $i\in\{1,2,\ldots,n\}$ we have

$$(a_1^{i-1},\sum_{j=1}^{m} b_j, a_{i+1}^{n})_\circ =$$

$$= \varphi_{(s+1)}(a_1,\alpha(a_2),\ldots,\alpha^{i-1}(\sum_{j=1}^{m} b_j),\alpha^{i}(a_{i+1}),\ldots,\alpha^{n-1}(a_n),c_1^{k-1}) =$$

$$= \varphi_{(s+1)}(a_1,\alpha(a_2),\ldots,\sum_{j=1}^{m}\alpha^{i-1}(b_j),\alpha^{i}(a_{i+1}),\ldots,\alpha^{n-1}(a_n),c_1^{k-1}) =$$

$$= \sum_{j=1}^{m}\varphi_{(s+1)}(a_1,\alpha(a_2),\ldots,\alpha^{i-1}(b_j),\alpha^{i}(a_{i+1}),\ldots,\alpha^{n-1}(a_n),c_1^{k-1} =$$

$$= \sum_{j=1}^{m}(a_1^{i-1},b_j,a_{i+1}^{n})_\circ .$$

which proved that the distributive laws hold in $(R,+,\circ)$.

By proposition 2 and Theorem 3 [9] result:

COROLLARY *If* $(R,+,\psi)$ *is an (m,k)-division ring,* $n=(k-1)s+1$; $s\in\mathbb{N}^*$, $\alpha\in End\ (R,+,\psi)$, $c_1,\ldots,c_{k-1}\in R$ *are fixed elements, then* $(R,+,\varphi) = ext^{(m,n)}_{\alpha,c_1^{k-1}}(R,+,\psi)$ *is an (m,n)-division ring if and only if* $\alpha\in Aut(R,+,\psi)$ *and the condition (5) holds.*

In sequel, in the special case of (m,n)-rings with right unit is proved the following

THEOREM *If* $(u_1,\ldots,u_{n-1})$ *is a right unit in the (m,n)-ring* $(R,+,\psi)$; $(R,+,\varphi) = \mathop{red}\limits^{(m,k)}_{u_1^{s-1}}(R,+,\psi)$ $n-1=s(k-1)$; $k\geq 2$ *and*

(6) $\alpha: R \to R$; $\alpha(x) = \psi(u_s^{n-1} x\, u_1^{s-1})$.

(7) $c_1^* = \overset{(n)}{\psi}_{(n+1-s)}(u_s^{n-1}, u_1^s)$;

(8) $c_i = \psi(u_s^{n-1}, u_{(i-1)s+1}^{is})$; $i=\overline{1,k-1}$,

then $(R,+,\varphi)$ is an (m,k)-ring with the right unit $c_1, c_2 \ldots c_{k-1}$; $\alpha \in End(R,+,\varphi)$ and

(9) $ext^{(m,n)}_{\alpha; c_1^* c_2^{k-1}}(red^{(m,k)}_{u_1^{s-1}}(R,+,\psi)) = (R,+,\psi)$

Proof. By proposition 1, the algebraic system $(R,+,\varphi)$, where $\varphi(x_1^k) = \psi(x_1, u_1^{s-1}, x_2, u_1^{s-1}, \ldots, x_k)$ is an (m,k)-ring. Because

$$\varphi(xc_1^{k-1}) = \psi(x, u_1^{s-1}, c_1, u_1^{s-1}, c_2, \ldots, u_1^{s-1}, c_{k-1}) =$$

$$= \psi(x, u_1^{s-1}, \psi(u_s^{n-1}, u_1^s)\, u_1^{s-1}, \psi(u_s^{n-1}, u_{s+1}^{2s}), u_1^{s-1}, \ldots$$

$$\ldots, u_1^{s-1}, \psi(u_s^{n-1}, u_{(k-2)s+1}^{(k-1)s}))$$

and by associativity laws we have

$$\varphi(xc_1^{k-1}) = \psi(x, u_1^{n-1}, u_1^s, u_1^{n-1}, u_{s+1}^{2s}, \ldots, u_1^{n-1} u_{(k-2)s+1}^{n-1}) =$$

$$= \psi(x, u_1^s, u_{s+1}^{2s}, \ldots, u_{(k-2)s+1}^{n-1}) = \psi(x, u_1^{n-1}) = x$$

for all $x \in R$, results that c_1^{k-1} is a right unit for (m,k)-ring $(R,+,\varphi)$.

Because the operation ψ is distributive relatively to "+", for all $x_1, x_2, \ldots, x_m, y_1, \ldots, y_k \in R$ we have

$$\alpha(\sum_{i=1}^{m} x_i) = \psi(u_s^{n-1}, \sum_{i=1}^{m} x_i, u_1^{s-1}) = \sum_{i=1}^{m} \psi(u_s^{n-1}, x_i, u_1^{s-1}) = \sum_{i=1}^{m} \alpha(x_i)$$

By hypothesis u_1^{n-1} is a right unit in the n-semigroup (R,ψ);

then using the associativity laws we have:

$$\alpha(\varphi(y_1^k)) = \psi(u_s^{n-1}, \varphi(y_1^k), u_1^{s-1}) =$$

$$= \psi(u_s^{n-1}, \psi(y_1, u_1^{s-1}, y_2, u_1^{s-1}, \ldots, u_1^{s-1}, y_k), u_1^{s-1}) =$$

$$= \psi_{(\cdot)}(u_s^{n-1}, y_1, u_1^{s-1}u_1^{n-1}, y_2, u_1^{n-1}, \ldots, u_1^{n-1}, y_k, u_1^{s-1} =$$

$$= \psi(\psi(u_s^{n-1}y_1u_1^{s-1}, u_1^{s-1}, \psi(u_s^{n-1}, y_2)u_1^{s-1}), u_1^{s-1}, \ldots, u_1^{s-1}\psi(u_s^{n-1}y_ku_1^{s-1})) =$$

$$= \psi(\alpha(y_1), u_1^{s-1}, \alpha(y_2), u_1^{s-1}, \ldots, u_1^{s-1}, \alpha(y_k)) =$$

$$= \varphi(\alpha(y_1), \alpha(y_2), \ldots, \alpha(y_k)) \ .$$

This proved that $\alpha \in End(R,+,\varphi)$.

Analogous with the proof of the theorem which generalized the Zupnik's Theorem [8] is verified that the condition (5) is fulfilled for $c_1^*, c_2, \ldots, c_{k-1}$ defined by (7) and (8). Therefore the extension $ext^{m,n}_{\alpha, c_1^* c_2^{k-1}}(R,+,\varphi) = (R,+,\circ)$ is an (m,n)-ring.

But, by the definitions of the aplication α and of the operations "$\circ$" and "φ" we have

$$(x_1^n)_\circ = \varphi_{(s+1)}(x_1, \alpha(x_2), \ldots, \alpha^{n-1}(x_n), c_1^*, c_2^{k-1}) =$$

$$= \varphi_{(s)}(\psi(x_1, u_1^{s-1}, \alpha(x_2), u_1^{s-1}, \ldots, u_1^{s-1}\alpha^{n-1}(x_n)u_1^{s-1}, c_1^*u_1^{s-1}c_2 \ldots u_1^{s-1}c_{k-1})) =$$

$$= \varphi_{(\cdot)}(\psi_{(\cdot)}(x_1, x_2\overset{(2)}{u_1^{s-1}}\overset{(2)}{u_s^{n-1}}x_3 \ldots x_n\overset{(n)}{u_1^{s-1}}\overset{(n)}{u_s^{n-1}}u_1^{s}u_1^{s-1}u_s^{n-1}u_{s+1}^{2s} \ldots u_1^{s-1}u_s^{n-1}, u_{n-s}^{n-1})) =$$

$$\ldots = \psi(\psi(x_1, x_2, x_3, \ldots, x_n)u_1^{s}u_{s+1}^{2s} \ldots u_{n-s}^{n-1})) = \psi(x_1^n) \ .$$

This proved that $(R,+,\circ) = (R,+,\psi)$.

In the speciale case k=2 we have the following corollary

COROLLARY 1. *If $(u_1, \ldots, u_{n-1})$ is a right unit in the (m,n)-ring $(R,+,\psi)$, then the (m,2)-reduce $(,+,\circ)$ relatively to $u_1, \ldots, u_{n-2}$ has the right unit u_{n-1} and there is an endomorphism α of $(R,+,\circ)$; $\alpha(x) = \psi(u_{n-1}, x, u_1^{n-2})$ and the element $c = \psi(u_{n-1}^{(n)}) = u_{n-1}^{[1]}$ such that*

$$ext^{(m,n)}_{\alpha; c}(red^{(m,2)}_{u_1^{n-2}}(\mathbf{R},+,\psi)) = (\mathbf{R},+,\psi)$$

COROLLARY 2. *To every right unit* $(u_1,\dots,u_{n-1})\in R^{n-1}$ *of an* (m,n) *division ring* $(R,+,\psi)$ *and to every number* $s\in N^*$ *defined by* $n-1=$ $=s(k-1)$; $k\in N$; $k\geq 2$, *there is an automorphism* $\alpha\in Aut(R,+,\varphi)$;

$\alpha(x) = \psi(u_s^{n-1}xu_1^{s-1})$; *where* $(R,+,\varphi) = red_{u_1^{s-1}}^{(m,k)}(R,+,\psi)$ *and elements*

$c_1 = \psi_{(n-s)}(\overset{(n-1)}{u_s^{(n-1)}}, u_s)$ *and* $c_i = \varphi(u_s^{n-1}, u_{(i-1)s+1}^{is}$; $i = \overline{i=2,k-1}$ *such that* $(R,+,\varphi)$ *is an* (m,k)*-division ring, and its* (m,n)*-extension relatively to* α *and* $c_1,\dots,c_{k-1}$ *is the* (m,n)*-division ring* $(R,+,\psi)$*, that is*

$$ext_{\alpha,c_1^{k-1}}^{(m,n)}(red_{u_1^{s-1}}^{(m,k)}(R,+,\varphi) = (R,+,\varphi)$$

From corollary 1 and 2 and by the corollary of proposition 2 we have

COROLLARY 3. *If* $(R,+,\psi)$ *is an* (m,n)*-division ring* $a\in R$*, then there exist* $c\in R$; $c=a^{[1]}$; *and the* $(m,2)$- *division ring*

$(R,+,\circ) = red_a^{(m,2)}(R,+,\psi)$ *with unit "a" and* $\alpha\in Aut(R,+,\circ)$ *where* $\alpha(x)=\psi(a,x,a,\dots\underline{a})$ *such that*

$(R,+,\psi) = ext_{\alpha,c}^{(m,n)}(R,+,\circ)$.

The present theorem and corollaries allow the construction of generalized ring starting from a given (m,n)-ring.

<u>Remark</u>. The related reduces and extensions of an (m,n)-ring were born from the modification of multiplicative operation. It is easily seen that same kind of change of additive operation implies, for to preserve the distributive laws, the existence of zero element in the given (m,n)-ring.

ABSTRACT.

In this paper, some extensions and reduces of (m,n)-rings are defined through the change of n-semigroup operation, and the connection between them in the case of unitary (m,n)-rings is studies.

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