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ON (m,n) - GENERALIZED RINGS

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Summary. In the papers [1],[2],[4],[6],[9] various authors continue the study of ordinary rings to the case where the underlying group and semigroup are respectively an m-ary commutative group and an n-ary semigroup. Because the usual commutative group concept may by generalized also as semicommutative m-group (by Dörnte [3]), the following paper is concerned with the extension in this sense of the usual (m,n)-ring concept.

For self-containment we give some definitions and results which will be used in the sequel.

1. NOTIONS AND PRELIMINARY RESULTS

Definition 1.1. An <u>n-semigroup</u> is an algebraic system $(A,\circ)$ with one n-ary operation $\circ: A^n \to A$, $n\in\mathbb{N}$, $n\geq 2$ such that for any set of elements $a_1,a_2,\ldots,a_{2n-1}\in A$ and any $k=1,\ldots,n-1$ it is true that

$$((a_1,\ldots,a_n)_\circ, a_{n+1},\ldots,a_{2n-1})_\circ=$$
$$=(a_1,\ldots,a_k,(a_{k+1},\ldots,a_{k+n})_\circ, a_{k+n+1},\ldots,a_{2n-1})_\circ$$

shortly $\left(\left(a_1^n\right)_\circ a_{n+1}^{2n-1}\right)_\circ=\left(a_1^k\left(a_{k+1}^{k+n}\right)_\circ a_{k+n+1}^{2n-1}\right)_\circ$

Definition 1.2. An <u>n-group</u> is an n-semigroup $(A,\circ)$ in which the equations $(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n)_\circ = a_i$ have a unique solution in A for arbitrary $a_1,\ldots,a_n\in A$ and for each $i\in\{1,\ldots,n\}$.

Definition 1.3. An n-semigroup (n-group) $(A,\circ)$ is <u>commutative</u> if the operation "$\circ$" is invariant under each permutation of the elements involved.

Definition 1.4. [3] An n-semigroup (n-group) is semicommutative if $(a_1,a_2,\ldots,a_{n-1},a_n)_\circ=(a_n,a_2,\ldots,a_{n-1},a_1)_\circ$ for arbitrary $a_1,a_2,\ldots,a_n\in A$.
Evidently, for n=2 the commutative and semicommutative n-semigroup (n-group) concepts coincide.

Definition 1.5. [5] An n-semigroup (A,∘) is entropic (medial, for other authors) if

$$((a_{11},a_{12},\ldots,a_{1n})_\circ,(a_{21},a_{22},\ldots,a_{2n})_\circ,\ldots,(a_{n1},a_{n2},\ldots,a_{nn})_\circ)_\circ=$$

$$=((a_{11},a_{21},\ldots,a_{n1})_\circ,(a_{12},a_{22},\ldots,a_{n2})_\circ\ldots(a_{1n},a_{2n},\ldots,a_{nn})_\circ)_\circ$$

for arbitrary $a_{ij}\in A; i,j\in\{1,\ldots,n\}$.

Definition 1.6. An element a∈A of an n-semigroup (A,∘) is called idempotent if $(a,a,\ldots,a)_\circ=a$.

Definition 1.7. An element e∈A is called an i-identity (identity) element of (A,∘) if for each x∈A we have

$$(e,\ldots,e,x,e,\ldots,e)_\circ=x\ ((x,e,\ldots,e)_\circ=(e,x,\ldots,e)_\circ=(e,\ldots e.x)_\circ=x)$$

Definition 1.8. In the n-group (A,∘), the solution of the equation $(a,a,\ldots,a,x)_\circ=a$ is called the querelement of "a" (by Dörnte [3]) and it is denoted by $\bar{a}$. The element $\bar{a}$ has the additional property

$$(x,a,\ldots,\bar{a},\ldots,a)_\circ=(a,\ldots,\bar{a},\ldots,a,x)_\circ=x$$

for each x∈A.

Proposition 1.1. [3] If (A,∘) is a semicommutative n-semigroup then it is an entropic n-semigroup.
The converse is not true, for example $(A,\circ);(a_1,\ldots,a_n)_\circ=a_1$ $\forall\, a_1,\ldots,a_n\in A$ is an entropic n-semigroup but not a semicommutative one.

Proposition 1.2. If (A,∘) is an entropic n-group then (A,∘) is semicommutative.

Proof. For each $a_1,\dots,a_n \in A$ we have

$$(a_1,a_2,\dots,a_{n-1}a_n)_\circ=((a_1,a_2,\dots,a_n)_\circ,\overline{a}_n,a_n,\dots,a_n)_\circ=$$

$$=((a_1,a_2,\dots,a_n)_\circ,(\overline{a}_1,a_1,\dots,a_1,\overline{a}_n)_\circ,(a_1,\overline{a}_1,\dots,a_1,a_n)_\circ,\dots$$

$$\dots(a_n,a_1,\dots,\overline{a}_1,a_1)_\circ)_\circ=(\textit{by entropy})=$$

$$=((a_1,\overline{a}_1,a_1,\dots,a_1,a_n)_\circ(a_2,a_1,\overline{a}_1,\dots,a_1)_\circ,\dots,(a_{n-1},a_1,\dots,\overline{a}_1)_\circ,$$

$$,a_n,\overline{a}_n,\dots,a_n,a_1)_\circ)_\circ=(a_n,a_2,\dots,a_{n-1},a_1)_\circ$$

Corollary 1.1. If $(A,\circ)$ is a semicommutative n-group, then $\overline{(a_1,\dots,a_n)_\circ}=(\overline{a}_1,\overline{a}_2,\dots,\overline{a}_n)_\circ \ \forall a_1,\dots,a_n \in A$.

Proof. By proposition 1.2 we have

$$((a_1,\dots,a_n)_\circ,\dots(a_1,\dots,a_n)_\circ,(\overline{a}_1,\dots,\overline{a}_n)_\circ)_\circ=$$

$$=((a_1,\dots,a_1,\overline{a}_1)_\circ,\dots,(a_n,\dots,a_n,\overline{a}_n)_\circ)_\circ=$$

$= (a_1,\dots,a_n)_\circ$, $\forall a_1,\dots,a_n \in A$, hence by definition 1.8

$\overline{(a_1,\dots,a_n)_\circ} = (\overline{a}_1,\dots,\overline{a}_n)_\circ$.

Definition 1.9. Let $(A,\circ)$ be an n-group. A non empty subset B of A is called a <u>sub-n-group</u> of $(A,\circ)$ if the restriction of "$\circ$" to B makes it an n-group.

Proposition 1.3. A non empty subset B of A is a sub-n-group of the n-group $(A,\circ)$ if and only if:

1° $x_1,x_2,\dots,x_n \in B \rightarrow (x_1,x_2,\dots,x_n)_\circ \in B$;

2° $x \in B \rightarrow \overline{x} \in B$.

Definition 1.10. A subset $I \subseteq A$ of an n-semigroup $(A,\circ)$ is an <u>i-ideal</u>, $i \in \{1,2,\dots n\}$, of A if $(A,\dots,A,\underset{i}{I},A,\dots,A)_\circ \subseteq I$. An i-ideal of A for all $i=1,2,\dots,n$ is called <u>ideal</u> of $(A,\circ)$.

Definition 1.11. An element $z \in A$ is called a <u>zero</u> of A if $(z,x_1,\dots,x_{n-1})_\circ=(x_1,z,\dots,x_{n-1})_\circ=\dots=(x_1,\dots,x_{n-1},z)_\circ=z$, for every $x_1,\dots,x_{n-1}\in A$.

Definition 1.12. An equivalence relation ρ of an n-semigroup (A,∘) is called a <u>congruence</u> of the n-semigroup if the relation is compatible with the n-ary operation "∘"; by this we mean

$$\forall x_i,y_i\in A\ ;\ x_i\rho y_i\ ;\ i=1,\dots,n \rightarrow (x_1,\dots,x_n)_\circ\rho(y_1,\dots,y_n)_\circ$$

In $A/\rho=\{\hat{a}/a\in A\};\ \hat{a}=\{x\in A/a\rho x\}$ we define an n-ary operation "*" by

$(\hat{x}_1,\hat{x}_2,\dots,\hat{x}_n)_*=(x_1,x_2,\dots,x_n)_\circ$.

It is easily seen that

Proposition 1.4. [7] 1° If (A,∘) is a semicommutative n-group and ρ is a congruence of (A,∘), then (A/ρ,*) is a semicommutative n-group, where the querelement of the equivalence class $\hat{a}$ is $\hat{\bar{a}}$;

2° The equivalence class $\hat{a}$ is a sub-n-group (A,∘) if and only if the element a∈A is 1-identity for (A,∘).

3° If H is a sub-n-group of the semicommutative n-group (A,∘), then there is a unique congruence ρ

$$a\rho b \leftrightarrow (aH\dots H)_\circ=(bH\dots H)_\circ$$

so that $H\in A/\rho$ and $A/\rho\simeq A/H;\ A/H=\{(xH\dots H)_\circ;\ x\in A\}$.

4° The equivalence class $\hat{a}$ is an ideal of the n-semigroup (A,∘) if and only if $\hat{a}$ is a zero in A/ρ.

2. GENERAL DEFINITIONS AND ELEMENTARY PROPERTIES

Definition 2.1. An universal algebra (R,[],∘); $[]:\ R^m\rightarrow R$; $\circ:R^n\rightarrow R$ is an <u>(m,n)-generalized ring</u>, $m,n\in N^*\setminus\{1\}$, if:

1° (R,[]) is a semicommutative m-group;

2° (R,∘) is an n-semigroup;

3° the following distributive laws hold for all choices of $a_1,a_2,\dots,a_n,b_1,\dots,b_m\in R$ and for all choices of $i\in\{1,2,\dots,n\}$:

$$(a_1,\dots,a_{i-1},[b_1,\dots,b_m],a_{i+1},\dots,a_n)_\circ=$$

$$=[(a_1,\dots,a_{i-1},b_1,a_{i-1},\dots,a_n)_\circ,\dots,(a_1,\dots,a_{i-1},b_m,a_{i+1},\dots,a_n)_\circ] .$$

Clearly, an ordinary ring is a (2,2)-generalized ring.

Example 2.1. $(\mathbb{R},[\,],\circ)$ where $\mathbb{R}$ is the set of real numbers

$$[\,]:\mathbb{R}^{2m+1}\to\mathbb{R};\ [x_1,x_2,\ldots,x_{2m+1}]=x_1-x_2+x_3-\ldots+x_{2m+1}$$

$$\circ:\mathbb{R}^{n}\to\mathbb{R};\ (x_1,x_2,\ldots,x_n)_\circ=x_1$$

is a $(2m+1,n)$ - generalized ring.

Example 2.2. $(Z_n,[\,],\circ)$ where $[\hat{a},\hat{b},\hat{c}]=\hat{a}+\widehat{n-b}+\hat{c}$, $\hat{a}\hat{b}=\widehat{ab}$ is a (3,2)-generalized ring.

As in ordinary (m,n)-rings [1], we define <u>the semiadditive idempotent</u> and <u>the multiplicative idempotent</u> as an idempotent element in (R,[]) respectively in (R,∘).

The element $\overline{a}$ will denote the semiadditive querelement of a in the m-group (R,[]) and the element $\underline{a}$ will denote - if it exists - the multiplicative querelement of "a" in the n-semigroup (R,∘).

Definition 2.2. An element $z\in R$ is called a <u>zero</u> of R if it is a zero of the n-semigroup (R,∘).

Evidently, if it exists, a zero of R is a multiplicative and semiadditive idempotent. A (m,n)-generalized ring may have at most one zero. A semiadditive and multiplicative idempotent is not necessarily a zero element; in example 2.1, every $a\in R$ is a semiadditive and multiplicative idempotent, but this (2m+1,n)-generalized ring has not a zero element.

In the example 2.2, every element of Z_n is a semiadditive idempotent, $\hat{0}$ and $\hat{1}$ are multiplicative idempotents, and $\hat{0}$ is a zero element.

Proposition 2.1. If (R,[],∘) is an (m,n)-generalized ring, $a_1,a_2,\ldots,a_n\in R$, then

$$\overline{(a_1,a_2,\ldots,a_n)_\circ}=(a_1,a_2,\ldots,\overline{a}_i,\ldots,a_n)_\circ\ ;\ i=1,2,\ldots,n$$

Proof. Because (R,[]) is an m-group, by querelement's definition and by distributive law we have

$$(a_1,\ldots,a_i,\ldots,a_n)_\circ=(a_1,\ldots,[a_i,\ldots,a_i,\overline{a}_i],\ldots,a_n)_\circ=$$

$=[(a_1,\dots,a_i,\dots,a_n)_\circ,\dots,(a_1,\dots,a_i,\dots,a_n)_\circ,(a_1,\dots,\overline{a}_i,\dots,a_n)_\circ]$

whence $\overline{(a_1,\dots,a_i,\dots,a_n)}_\circ=(a_1,\dots,\overline{a}_i,\dots,a_n)_\circ$.

Proposition 2.2. 1° If $a_1,\dots,a_m\in R$ are semiadditive idempotents of the (m,n)-generalized ring $(R,[],\circ)$, then $[a_1,\dots,a_m]$ is an semiadditive idempotent too;

2° If $a\in R$ is an semiadditive idempotent then for every $b_1,\dots,b_n\in R$ and for each $i\in\{1,\dots,n\}$ the element $(b_1,\dots,b_{i-1},a,b_{i+1},\dots,b_n)_\circ$ is an semiadditive idempotent too.

Proof. Indeed, by proposition 1.1 the operation [] is entropic, and

1° $[[a_1,\dots,a_m],\dots,[a_1,\dots,a_m]]\overset{ent}{=}[[a_1,\dots,a_1],\dots,[a_m,\dots,a_m]]=$

$=[a_1,\dots,a_m]$

2° $(b_1,\dots,b_{i-1}ab_{i+1},\dots,b_n)_\circ=(b_1,\dots,b_{i-1},[a,\dots,a]b_{i+1},\dots,b_n)_\circ=$

$[(b_1..b_{i-1}ab_{i+1},\dots,b_n)_\circ,\dots,(b_1,\dots,b_{i-1},a,b_{i+1},\dots,b_n)_\circ]$.

Definition 2.3. A subset $S\subseteq R$ is called an <u>(m,n)-subring</u> of$(R,[],\circ)$ if S is a sub-m-group of $(R,[])$ and S is closed for the n-ary operation of R; this means

$[S,S,\dots,S]=S$, $\overline{S}\subseteq S$,where $\overline{S}=\{\overline{x}\in R|x\in S\}$ and $(S,S,\dots,S)_\circ\subseteq S$.

It is easy to check that the set $(R,[],\circ)$ of the (m,n)-subrings of the generalized (m,n)-ring $(R,[],\circ)$ is an algebraic closure system. Consequently, $(\mathcal{S}(R,[],\circ),\subseteq)$ is an algebraic lattice.

Definition 2.4. A subset $I\subseteq R$ is called an <u>i-ideal</u>, $i\in\{1,2,\dots,n\}$ of R if $(I,[])$ is a sub-m-group of $(R,[])$ and $(R^{i-1}IR^{n-i})_\circ\subseteq I$. If I is an i-ideal of R for all $i=1,2,\dots,n$, then I is called <u>ideal</u> of $(R,[],\circ)$.

The following properties of (m,n)-rings (see [1],[4]) remain valid in the case of (m,n)-generalized rings:

Proposition 2.3. i) The intersection of an arbitrary number of i-ideals of R is an i-ideal, too.

ii) If $I_1, I_2, \ldots, I_n$ are i-ideals of R, then $[I_1, I_2, \ldots, I_n]$ is an i-ideal of R.

iii) Let $B_1, B_2, \ldots, B_n$ be subsets of a ring R and define

$$\langle (B_1, B_2, \ldots, B_n)_\circ \rangle = \{[(b_{11}, b_{12}, \ldots, b_{1n})_\circ, \ldots, (b_{p1}, b_{p2}, \ldots, b_{pn})_\circ] \mid b_{ij} \in B_j, \; p \equiv 1 \,(\mathrm{mod}\; m-1)\}$$

If R is a commutative (m,n) generalized ring and one subset (say B_i) is an ideal of R then $\langle (B_1, B_2, \ldots, B_n)_\circ \rangle$ is an ideal too.

iv) I is an ideal of R if and only if $\overline{I}$ is an ideal

v) If I is an ideal of R then $\tilde{I} = \{x \in R \mid \overline{x} \in I\}$ is an ideal, too.

Proof. Easy by corollary 1.1 and proposition 2.1.

Definition 2.5. By an <u>i-center</u> ($i \in \{2, \ldots, n\}$) of an (m,n)-generalized ring we mean the set

$$C_i(R) = \{a \in R \mid (a x_2^n)_\circ = (x_2^i, a, x_{i+1}^n)_\circ, \; \forall x_2, \ldots, x_n \in R\} .$$

$C(R) = \bigcap_{i=2}^{n} C_i(R)$ is called the <u>center</u> of an (m,n)-generalized ring.

Proposition 2.4. If $C_i(R)$ is non-empty, then it is an (m,n)-subring of $(R, [\,], \circ)$.

Proof. Corollary 1.1. and proposition 2.1. allow a proof analogous to the one given by Dudek in [4].

Corollary 2.1. i) If C(R) is non-empty, then it is a maximal commutative (m,n)-subring of R.

ii) An (m,n) generalized ring R is commutative if and only if $C(R) = R \Leftrightarrow C_2(R) = C_n(R) = R$.

Definition 2.6. Let R and R′ be (m,n)-generalized rings. A mapping $f: R \to R'$ is called <u>homomorphism</u> if $f([x_1, \ldots, x_m]) =$

$= [f(x_1), \ldots, f(x_m)]$ and $f((x_1, \ldots, x_n)_\circ) = (f(x_1), \ldots, f(x_n))_\circ$,

$\forall x_i \in R, \; i = 1, \ldots, \max(m, n)$.

Proposition 2.5. Let $(R, [\,], \circ)$, $(R' [\,], \circ)$ be (m,n)-generalized rings and $f: R \to R'$ a homomorphism. The following properties are immediate:

i) Each semiadditive idempotent of R is mapped by f in a semiadditive idempotent of R′;

ii) $f(\overline{x})=\overline{f(x)}$;

iii) If f is onto and R has a zero 0, then f(0)=0′ is a zero in R′;

iv) If S is an (m,n)-subring of R, then f(S) is an (m,n)-subring of R′;

v) If f is onto and S′ is an (m,n)-subring of R′, then $f^{-1}(S')$ is an (m,n)-subring of R;

vi) If f is onto and I is an i-ideal of R, then f(I) is an i-ideal of R′;

vii) If f is onto and I′ is an i-ideal of R′, then $f^{-1}(I')$ is an i-ideal of R.

Definition 2.7. If f is a homomorphism of a ring R onto a ring R′having a zero 0′,we call <u>kernel of f</u> the set Kerf={x∈R| f(x)=0′}.

Proposition 2.6. The kernel Ker f is an ideal of R.

Proof. This proposition is a consequence of proposition 2.5, (vii)

Proposition 2.7. If I is an ideal in a ring R, then R/I with addition defined by proposition 1.4. and with multiplication defined by

$$([x_1,I,\ldots,I],\ldots,[x_n,I,\ldots,I])_\circ=[(x_1,\ldots,x_n)_\circ,I,\ldots,I]$$

is a ring.

Corollary 2.2. The binary relation defined by aρb ⇔ [a,I,..,I]= = [b,I,...,I] is a congruence of R such that I∈R/ρ and R/ρ≃R/I.
Note that u∈I ⇔ [u,I,...,I] = I.
If R has a zero 0, this zero belongs to all i-ideals and also to each ideal. Moreover, the subset of R consisting of 0 alone is an ideal denoted by (0) and called the <u>zero-ideal</u>. It is the only ideal of R consisting of one element.

Proposition 2.8. If R has a zero 0 and ρ is a congruence of R then ρ<0> is an ideal of R, and every ideal of R can be regarded as the congruence class of 0 with respect to some congruence of R.

Proof. If 0 is a zero in R, then $\rho<0>$ is a zero in R/ρ and by proposition 1.4., 4", $\rho<0>$ is an ideal of $(R,\circ)$; $\rho<0>$ being a sub-m-group of $(R,[])$ we get that $\rho<0>$ is an ideal of the (m,n) generalized ring $(R,[],\circ)$.

If I is an ideal of R, by $0\in I$ it follows that $I=\rho<0>$ (where ρ is the congruence defined by corollary 2.2).

3. ON (m,2)-REDUCED RINGS AND (m,2)-ASSOCIATED RINGS OF AN (m,n)-GENERALIZED RING

Let $(R,[],\circ)$ be an (m,n) generalized ring and $u_1,\ldots,u_{n-2}$ fixed elements of R. Define a binary operation on R

$\cdot : RxR\to R$ by $x\cdot y=(x,u_1^{n-2},y)_\circ$. It is easily verified that this operation is associative and distributive with respect to the m-ary operation [], hence $(R,[],\cdot)$ is an (m,2) generalized ring.

Definition 3.1. If $(R,[],\circ)$ is an (m,n) generalized ring, then $(R,[],\cdot)$ is an (m,2) generalized ring called the <u>reduced ring</u> with respect to the elements $u_1,\ldots,u_{n-2}\in R$ and denoted by $red_{u_1^{n-2}}(R,[],\circ)$.

Remark 3.1. If $u_1,u_2,\ldots,u_{n-1}$ shortly u_1^{n-1} , is a right unit (as a system of n-1 elements) in $(R,[],\circ)$, then u_{n-1} is a right unit in $red_{u_1^{n-2}}(R,[],\circ)$. The element u_{n-1} is a unit in the (m,2) reduced ring $red_{u_1^{n-2}}(R,[],\circ)$ if and only if u_1^{n-1} is a right unit and $u_{n-1}u_1^{n-2}$ is a left unit in the (m,n) generalized ring.

Example 3.1. If $(Z_4,[],\circ)$ is the commutative (3,3) generalized ring, where $[\hat{x},\hat{y},\hat{z}]=\hat{x}+\widehat{4-y}+\hat{z}$ and $(\hat{x},\hat{y},\hat{z})_\circ=\hat{x}\cdot\hat{y}\cdot\hat{z}$ then $red_{\hat{3}}(Z_4,[],\circ)$ is a commutative (3,2) generalized ring with the unit $\hat{3}$.

The multiplication table is

$\cdot$	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{0}$
$\hat{1}$	$\hat{3}$	$\hat{2}$	$\hat{1}$	$\hat{0}$
$\hat{2}$	$\hat{2}$	$\hat{0}$	$\hat{2}$	$\hat{0}$
$\hat{3}$	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{0}$
$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$

In the same manner $red_2(Z_4,[],\circ)$ is a commutative (3,2) generalized ring and the multiplication table is

$*$	$\hat{0}$	$\hat{1}$	$\hat{2}$	$\hat{3}$
$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$
$\hat{1}$	$\hat{0}$	$\hat{2}$	$\hat{0}$	$\hat{2}$
$\hat{2}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$
$\hat{3}$	$\hat{0}$	$\hat{2}$	$\hat{0}$	$\hat{2}$

We shall give now a construction of an (m,2) generalized ring on a covering set of R.

Define on $R^{n-1}=\{a_1^{n-1} \mid a_i\in R, i=\overline{1,n-1}\}$ a binary relation ρ by:

$a_1^{n-1}\rho b_1^{n-1} \Leftrightarrow (xa_1^{n-1})_\circ = (xb_1^{n-1})_\circ, \forall x\in R$. ρ is an equivalence relation; denote the equivalence class of a_1^{n-1} with $\langle a_1^{n-1}\rangle$. On the factor set $R^{n-1}/\rho \overset{not}{=} R_*$ define a binary operation "*" by:

$$\langle a_1^{n-1}\rangle * \langle b_1^{n-1}\rangle = \langle a_1^{n-2}(a_{n-1}b_1^{n-1})_\circ\rangle$$

It is easily verified that this operation is well defined and it is associative, hence $(R_*,\cdot)$ is a semigroup.

If u_1^{n-1} is a right unit in the (m,n) generalized ring $(R,[],\circ)$ then the equivalence ρ defined above coincides with the relation "~" defined by:

$$a_1^{n-1}\sim b_1^{n-1} \Leftrightarrow (u_{n-1}a_1^{n-1})_\circ = (u_{n-1}b_1^{n-1})_\circ .$$

Indeed if $a_1^{n-1} \sim b_1^{n-1}$ then $\forall x \in R$ we have

$$(xa_1^{n-1})_\circ = ((xu_1^{n-1})_\circ a_1^{n-1}) = (xu_1^{n-2}(u_{n-1}a_1^{n-1}))_\circ =$$

$$= (xu_1^{n-2}(u_{n-1}b_1^{n-1})_\circ)_\circ = ((xu_1^{n-1})_\circ b_1^{n-1})_\circ = (xb_1^{n-1})_\circ \text{ , hence } a_1^{n-1}\rho b_1^{n-1} .$$

The converse is obvious.

This remark shows that $R_\circ = R^{n-1}/\sim$ and $(R_\circ, \cdot)$ is a semigroup with the right unit $\langle u_1^{n-1} \rangle$.

Define an m-ary operation on $R_\circ$, $(\,)_+ : R_\circ^m \to R_\circ$ by:

$$(\langle a_{11}^{1,n-1} \rangle, \ldots, \langle a_{mi}^{m,n-1} \rangle)_+ = \langle u_1^{n-2}, [(u_{n-1}, a_{11}^{1,n-1})_\circ, \ldots, (u_{n-1}, a_{mi}^{m,n-1})_\circ] \rangle$$

It is obvious that this operation is well defined, i.e. it does not depend on the choice of the representatives.

Theorem 3.1. *If $(R,[\,],\circ)$ is an (m,n) generalized ring with a right unit u_1^{n-1} , then*

1) $(R_\circ,(\,)_+,*)$ is an $(m,2)$ generalized ring with right unit $\langle u_1^{n-1} \rangle$, called the $(m,2)$ generalized ring associated to $(R,[\,],\circ)$.

2) If $u_{n-1}u_1^{n-2}$ is a left unit in the (m,n) generalized ring $(R,[\,],\circ)$, then $(R_\circ,(\,)_+,*)$ is isomorphic to $red_{u_1^{n-2}}(R,[\,],\circ)$.

Proof. 1) By the definition of the operation $(\,)_+$, by the distributive law for $(R,[\,],\circ)$ and by associativity of the operation $[\,]$ we deduce that the operation $(\,)_+$ is associative.

Semicommutativity of the operation $[\,]$ implies semicommutativity of the operation $(\,)_+$.

$\forall \langle a_1^{n-1} \rangle \in R_\circ$ the equation

$(\langle a_1^{n-1} \rangle, \ldots, \langle a_1^{n-1} \rangle, \langle x_1^{n-1} \rangle)_+ = \langle a_1^{n-1} \rangle$, has the solution

$\langle x_1^{n-1} \rangle = \langle a_1, \ldots, \overline{a_k}, \ldots, a_{n-1} \rangle, \forall k=1,\ldots,n-1$, which is in fact the querelement of $\langle a_1^{n-1} \rangle$.

So, we have $\langle\overline{a_1^{n-1}}\rangle=\langle a_1,\ldots,\overline{a_k},\ldots,a_{n-1}\rangle$, $\forall k=1,\ldots,n-1$.

$\forall\langle x_1^{n-1}\rangle\in R_*$ the following equality holds in R_*:

$$\left(\langle x_1^{n-1}\rangle,\langle a_1^{n-1}\rangle,\ldots,\langle a_1^{n-1}\rangle,\overline{\langle a_1^{n-1}\rangle}\right)_+=\langle x_1^{n-1}\rangle .$$

Indeed,

$$\left(\langle x_1^{n-1}\rangle,\langle a_1^{n-1}\rangle,\ldots,\langle a_1^{n-1}\rangle,\overline{\langle a_1^{n-1}\rangle}\right)_+=$$

$$=\left\langle u_1^{n-2},\left[(u_{n-1},x_1^{n-1})_\circ,(u_{n-1},a_1^{n-1})_\circ,\ldots,(u_{n-1},a_1^{n-1})_\circ,(u_{n-1},a_1,\ldots,\overline{a_k},\ldots,a_{n-1})_\circ\right]\right\rangle$$

$$=\left\langle u_1^{n-2},\left[(u_{n-1},x_1^{n-1})_\circ,(u_{n-1},a_1^{n-1})_\circ,\ldots,(u_{n-1},a_1^{n-1})_\circ,\overline{(u_{n-1},a_1^{n-1})_\circ}\right]\right\rangle=$$

$$=\left\langle u_1^{n-2},(u_{n-1},x_1^{n-1})_\circ\right\rangle=\langle x_1^{n-1}\rangle .$$

These remarks show that $(R_*,()_+)$ is a semicommutative m-group. By a direct computation one can easily verify the distributivity law for * and $()_+$, so $(R,()_+,*)$ is an (m,2) generalized ring.

2) The mapping $f:R_*\to R$, $f(\langle a_1^{n-1}\rangle)=(u_{n-1}a_1^{n-1})_\circ$ is obviously well defined and one-to-one.

$$f(\langle a_1^{n-1}\rangle*\langle b_1^{n-1}\rangle)=f(\langle a_1^{n-2},(a_{n-1}b_1^{n-1})_\circ\rangle)=$$

$$=(u_{n-1}a_1^{n-2}(a_{n-1},b_1^{n-1})_\circ)_\circ=(((u_{n-1}a_1^{n-1})_\circ u_1^{n-1})_\circ b_1^{n-1})=$$

$$=((u_{n-1}a_1^{n-1})_\circ,u_1^{n-2},(u_{n-1},b_1^{n-1})_\circ)_\circ=(u_{n-1}a_1^{n-1})_\circ\cdot(u_{n-1}b_1^{n-1})_\circ=$$

$$=f(\langle a_1^{n-1}\rangle)\cdot f(\langle b_1^{n-1}\rangle) .$$

$$f((\langle a_{11}^{1,n-1}\rangle,\ldots,\langle a_{m1}^{m,n-1}\rangle)_+)=(u_{n-1},u_1^{n-2},[(u_{n-1},a_{11}^{1,n-1})_\circ,\ldots,(u_{n-1},a_{m1}^{m,n-1})_\circ])_\circ=$$

$$=[(u_{n-1},a_{11}^{1,n-1})_\circ,\ldots,(u_{n-1},a_{m1}^{m,n-1})_\circ]=[f(\langle a_{11}^{1,n-1}\rangle),\ldots,f(\langle a_{m1}^{m,n-1}\rangle)] .$$

This shows that f is a one-to-one homomorphism of (m,2) generalized rings.

If $u_{n-1}u_1^{n-2}$ is a left unit, then $\forall y\in R$ we have

$y=\langle u_{n-1}u_1^{n-2}y\rangle_\circ=f(\langle u_1^{n-2}y\rangle)$, which shows that f is onto.

Example 3.2. Consider the (3,3) generalized ring $(Z_4,[],\circ)$ defined in example 3.1 and the system $\hat{3},\hat{3}$ which is a unit in $(Z_4,\circ)$. The (3,2) generalized ring associated with $(Z_4,[],\circ)$ will consist of the following elements

$Z_{4_\circ}=\{\langle\hat{1},\hat{k}\rangle \mid \hat{k}\in Z_4\}$, where

$<\hat{1},\hat{0}>=\{(\hat{0},\hat{0}),(\hat{0},\hat{1}),(\hat{1},\hat{0}),(\hat{0},\hat{2}),(\hat{2},\hat{0}),(\hat{0},\hat{3}),(\hat{3},\hat{0}),(\hat{3},\hat{0}),(\hat{2},\hat{2})\}$

$<\hat{1},\hat{1}>=\{(\hat{1},\hat{1}),(\hat{3},\hat{3})\}$

$<\hat{1},\hat{2}>=\{(\hat{1},\hat{2}),(\hat{2},\hat{1}),(\hat{3},\hat{2}),(\hat{2},\hat{3})\}$

$<\hat{1},\hat{3}>=\{(\hat{1},\hat{3}),(\hat{3},\hat{1})\}$.

The multiplication table is:

*	$<\hat{1},\hat{0}>$	$<\hat{1},\hat{1}>$	$<\hat{1},\hat{2}>$	$<\hat{1},\hat{3}>$
$<\hat{1},\hat{0}>$	$<\hat{1},\hat{0}>$	$<\hat{1},\hat{0}>$	$<\hat{1},\hat{0}>$	$<\hat{1},\hat{0}>$
$<\hat{1},\hat{1}>$	$<\hat{1},\hat{0}>$	$<\hat{1},\hat{1}>$	$<\hat{1},\hat{2}>$	$<\hat{1},\hat{3}>$
$<\hat{1},\hat{2}>$	$<\hat{1},\hat{0}>$	$<\hat{1},\hat{2}>$	$<\hat{1},\hat{0}>$	$<\hat{1},\hat{2}>$
$<\hat{1},\hat{3}>$	$<\hat{1},\hat{0}>$	$<\hat{1},\hat{3}>$	$<\hat{1},\hat{2}>$	$<\hat{1},\hat{1}>$

and the addition is defined by $(<\hat{1},\hat{a}>,<\hat{1},\hat{b}>,<\hat{1},\hat{c}>)_+=<\hat{1},\widehat{a-b+c}>$.

The mapping $f:(Z_{4_\circ},()_+,*)\to(Z_4,[],\cdot)$ defined by

$f(<\hat{1},\hat{k}>)=(\hat{3},\hat{1},\hat{k})_\circ=\hat{3}\cdot\hat{k}$ is a (3,2) ring isomorphism.

Remark 3.2. Theorem 3.1 is also valid in the particular case of (m,n) ordinary rings having right unit as a system of n-1 elements.

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