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ON SOME MAXIMUM PRINCIPLES

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Summary. In this paper we intended to evidentiate some maximum principles for the elliptic systems of partial differential equations.

For each of the forms of maximum principles we consider those elliptic systems which have appropriate structures.

1. We consider, in the bounded domain $D\subset\mathbb{R}^n$, thefollowing class of systems of partial differential equations

$$\sum_{i,j=1}^{m} A_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}+2\sum_{i=1}^{m}(B_i(x)+a_i(x)I)\frac{\partial u}{\partial x_i}-C(x)u=f(x)\,, \tag{1}$$

where $a_i\in C(\overline{D})$, $A_{ij}, B_i, C\in C(\overline{D}, M_n(\mathbb{R}))$ and $f, u\in C(D,\mathbb{R}^n)$.

We suppose that the system (1) is uniformly elliptic, i.e., there exists $\gamma\in\mathbb{R}_+^*$ such that for each $\tau\in\mathbb{R}^n, \lambda\in\mathbb{R}^m$ and each $x\in D$, we have

$$\sum_{i,j=1}^{m}(\tau^t A_{ij}(x)\tau)\lambda_i\lambda_j\geq\gamma\|\lambda\|^2\ . \tag{2}$$

Further, we suppose that the matrices $A_{ij}, i,j=\overline{1,m}$, and the regular solution $u\in C^2(D,\mathbb{R}^n)\cap C(\overline{D},\mathbb{R}^n)$ of the system (1) are such that there exists the functions $a_{ij}\in C(\overline{D})$, $i,j=\overline{1,m}$ for which the vector $(A_{ij}-a_{ij})p_{ij}$ is orthogonal to the vector u, that means the relation

(3) $$(u, A_{ij}p_{ij}) = a_{ij}(u, p_{ij})$$

holds, where $p_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$,and $(\cdot,\cdot)$ is the usual inner product of $\mathbb{R}^n$.

We denote B_i^t the transpose of matrix B_i, and $(\overline{a}_{ij})$ is the inverse matrix (a_{ij}).

We have

Theorem 1. *If the system (1) is uniformly elliptic in the domain D and there exists the constant $C_0>0$ such that for any regular solution u of the system (1) is fullfiled the inequality*

(4) $$(u, Cu) - \sum_{i,j=1}^{m} \overline{a}_{ij}(B_i^t u, B_j^t u) + \quad |u|^{-2} \sum_{i,j=1}^{m} \overline{a}_{ij}(u, B_i u)(u, B_j u) \geq C_0 |u|^2$$

then the relation

(5) $$\sup_{D} |u(x)| \leq \sup_{D} |u(x)| + \frac{1}{C_0} \sup_{D} |f(x)|$$

holds.

Proof. We prove that at any point $x_0 \in D$ for which $|u|$ has a maximum, we have

(6) $$C_0|u| \leq |f| \; .$$

At the point x_0 the following relations

(7) $$(u, p_i) = 0 \, , \; i = \overline{1,m} \; , \text{ where } \; p_i = \frac{\partial u}{\partial x_i} \; \text{ and}$$

(8) $$\sum_{i,j=1}^{m} [(u_i, p_{ij}) + (p_i, p_j)]\lambda_i \lambda_j \leq 0 \; , \text{ for each } \; \lambda \in \mathbb{R}^m \; ,$$

hold.

By the inner product of system (1) with u, and the relation (7) we obtain

(9) $$\sum_{i,j=1}^{m} (u, A_{ij}p_{ij}) + 2\sum_{i=1}^{m} (u, B_i p_i) - (u, Cu) = (u, f) \; .$$

From an algebraic result, using the relation (3) we get

$$(10)\qquad \sum_{i,j=1}^{m} a_{ij}(p_i,p_j)-2\sum_{i=1}^{m}(u,B_ip_i)+(u,Cu)\le |u||f| \ .$$

Let now consider the first member of relation (10) as a second degree polinom in n·m variables (the components of vectors p_i), and then we obtain that it is minimum when the variables are satisfying the conditions (7), that is

$$(u,Cu)-\sum_{i,j=1}^{m}\overline{a}_{ij}(B_i'u_i,B_j'u)+|u|^{-2}\sum_{i,j=1}^{m}(u,B_iu)(u,B_ju) \ .$$

Then from (10) and (4) we obtain (6), and hence the conclusion of theorem.

In a special case we have the following result.

Theorem 2. *If in the system (1) the matrices* B_i *are antisymetric, yhat is,* $B_i=-B_i'$, $i=\overline{1,m}$ *, then the conclusion of theorem 1 hold if the relation (4) is replaced by*

$$(11)\qquad (u,Cu)-\sum_{i,j=1}^{m}\overline{a}_{ij}(B_i'u,B_j'u)\ge C_*|u|^2 \ .$$

Proof. It is easily to see that if $B_i=-B'_i$ then $(u,B_iu)=0$, and so the relation (4) on reduce to (11).

Remark.1If $A_{ij}=a_{ij}I$ then the condition (3) is trivial and in this case on obtain the result of C.Miranda [4].

2. We give an application to an elliptic equation with complex coefficients.

We consider, in the domain D, the elliptic equation

$$(12)\qquad \sum_{j,k=1}^{m} a_{jk}\frac{\partial^2 v}{\partial x_j\partial x_k}+2\sum_{k=1}^{m} b_k\frac{\partial v}{\partial x_k}-cv=f,$$

where $a_{jk}=a_{jk}$ are real functions, and b_k,c,f and v are complex functions. We have

Theorem 3. *If the quadratic form* $\sum_{j,k=1}^{m} a_{jk}\lambda_j\lambda_k$ *is positive defined, for any* $x\in D$ *and there exists the constant* $c_*>0$ *such that the functions* b_k *and* c *satisfy the inequality*

$$(13)\qquad Re\,C(x)-\sum_{j,k=1}^{m}\overline{\frac{a}{jk}}(x)\,Im b_j(x)\,Im b_k(x)\geq c_*>0\,,$$

then for any regular solution $v\in C^2(D)\cap C(\overline{D})$ *of equation (12) hold the following relation*

$$(14)\qquad \sup_{\overline{D}}|v(x)|=\leq\sup_{\Gamma}|v(x)|+\frac{1}{C_*}\sup_{D}|f(x)|\,.$$

holds.

Proof. Because the elements of equation (12) are complex we denote

$$(15)\qquad v=u_1+iu_2\ ,\ b_k=b_{k1}+ib_{k2}\ ,\ c=c_1+ic_2\ ,\ f=f_1+if_2$$

Then we obtain a system of two equations

$$(16)\qquad \begin{cases}\sum_{j,k=1}^{m} a_{jk}\frac{\partial^2 u_1}{\partial x_j\partial x_k}+2\sum_{k=1}^{m} b_{k1}\frac{\partial u_1}{\partial x_k}-2\sum_{k=1}^{m} b_{k2}\frac{\partial u_2}{\partial x_k}-c_1u_1+c_2u_2=f_1\\ \sum_{j,k=1}^{m} a_{jk}\frac{\partial^2 u_2}{\partial x_j\partial x_k}+2\sum_{k=1}^{m} b_{2k}\frac{\partial u_2}{\partial x_k}+2\sum_{k=1}^{m} b_{k1}\frac{\partial u_2}{\partial x_k}-c_2u_1-c_1u_2=f_2\end{cases}$$

This system has the same form as the system in relation (1), where

$$(17)\qquad A_{jk}=a_{jk}I\ ,\ B_k=\begin{bmatrix}0 & -b_{k2}\\ b_{k2} & 0\end{bmatrix}\ ,\ c=\begin{bmatrix}c_1 & -c_2\\ c_2 & c_1\end{bmatrix}\ ,\ a_k=b_{k1}\ .$$

Because the matrices B_k are antisymetric we apply Theorem 2. Through direct calculation follows

$$(18)\qquad (u,Cu)=c_1|u|^2\ ,\ and\ \ (B_k^{\prime}u,B_j^{\prime}u)=b_{k2}b_{j2}|u|^2\ ,$$

therefore, from the inequality (11) we have:

(19) $$c_1-\sum_{j,k=1}^{m} a_{jk}(x)\, b_{j2}b_{k2}\geq c_0>0\ ,$$

which is the condition (13), and thus the theorem is proved.

Remark 2. Because in the conditions (13) does not appear $\operatorname{Re} b_k(x)$, the Theorem 3 is applicable also to the system with complex coefficients

(20) $$\sum_{j,k=1}^{m}\frac{\partial}{\partial x_j}\left(a_{jk}\frac{\partial v}{\partial x_k}\right)+2\sum_{k=1}^{m}b_k\frac{\partial v}{\partial x_k}-cv=f\ ,$$

where the functions a_{jk} are real, $a_{jk}\in C^1(D)$, and $b_k,c,f,v\in C(D)$, with complex values.

3. For a strong elliptic system of partial differential equations we give a maximum principle in integral form.

Let u be a n-dimensional vector which satisfies the strong elliptic system

(21) $$\frac{\partial^2 u}{\partial t^2}+\sum_{i=1}^{m}A_{i0}\frac{\partial^2 u}{\partial x_i\partial t}+\sum_{i,j=1}^{m}A_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}+c(x,t)\,u=0$$

where $x\in D\subset\mathbb{R}^n$ is a bounded domain, $0\leq t\leq T$, with $T>0$ fixed.

We suppose that

(22) $u(x,t)\equiv 0\ ,\ x\in\Gamma$ *(the boundary of D)*, $t\in[0,T]$,

(23) *the matrices* A_{i0},A_{ij} *are constant*, $A_{ij}=A_{ji},\ ij=\overline{1,m}$, and

(24) $C\in C(\overline{D}x[0,T],M_{mn}(\mathbb{R}))$,

such that the matrix C is negative defined in $\overline{D}x[0,T]$.

We prove that the integral $\int_D u^2(x,t)\,dx$ takes its maximum only on the boundary of interval [0,T]. We have

Theorem 4. *If u is a solution of system (21) and there are fulfiled the conditions (22), (23), (24) then the maximum value of*

$\int_D u^2(x,t)\,dx$ *is attained on (0,T).*

Proof. Let be $f(x)=\int_D u^2(x,t)\,dx$.

Then $f'(t)=2\int_D u(x,t)\,u_t(x,t)\,dx$ and

(25) $$f''(t)=2\int_D u^{2t}(x,t)\,dx+2\int_D u(x,t)\cdot u_{tt}(x,t)\,dx\ .$$

On the base of system (21), the second integral in relation (25) becomes

(26) $$\int_D u(x,t)\,u_{tt}(x,t)\,dx=-\int_D u\left(Cu+\sum_{i=1}^{m} A_{i0}\frac{\partial^2 u}{\partial x_i \partial t}+\sum_{i,j=1}^{m} A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}\right)dx\ .$$

Therefore, from the conditions (22), (23), the relation (26) becomes

(27) $$\int_D u(x,t)\,u_{tt}(x,t)\,dx= = -\int_D uCudx+\int_D\sum_{i=1}^{m} A_{i0}\frac{\partial u}{\partial t}\frac{\partial u}{\partial x_j}dx+\int_D\sum_{i,j=1}^{m} A_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx$$

Because the system (21) is strong elliptic and the matrix C is negative defined we deduce that

(28) $$f''(t)\geq 0\ ,\ 0\leq t<T\ .$$

Then the following inequality holds

(29) $$f(x)\leq f(0)+f(T)\,t\ ,\ t\in[0,T]\ ,$$

and thus we obtain the conclusion of theorem.

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