Buletinul Ştiinţific al Universităţii din Baia Mare
Seria B, Matematică-Informatică, vol.X(1994), 23-28

ON A BERNSTEIN TYPE OPERATOR

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Abstract. In this paper on extends to two variables a linear positive of Bernstein type introduced and investigated by Bleiman, Butzer and Hahn [3]. The corresponding operator $U_{m,n}$ defined on the space $C_B(D)$ of B-continuous functions on $D = [0,+\infty)\times[0,+\infty)$ is given at (2.5). On proves that if $f\in C_b(D)$ then $\lim\limits_{m,n\to\infty} U_{m,n}f = f$, uniformly on every compact $[0,a]\times[0,b]$.
Theorem 2.4 shows an evaluation of the order of approximation of the function $f\in C_b(D)$ by $U_{m,n}f$.

1.Preliminaries

The notion of the B-continuous function is that from [1]. The analogous of the Korovkin uniform approximation theorem is expressed in:

1.1. Theorem [2]: *Be $D\subseteq\mathbb{R}^2$ an compact and $(L_{m,n})_{m,n\in\mathbb{N}}$ a row of linears and positive operators wich transform the functions of $C_b(D)$ in functions of $\mathbb{R}^{\mathbb{R}^2}$. For each $f\in C_b(D)$ we note:*

$$U_{m,n}f(x,y):=L_{m,n}[f(\circ,y)+f(x,*)-f(\circ,*);x,y]$$

If:

i) $L_{m,n}(e;x,y)=1$; $e(s,t)=1$

ii) $L_{m,n}(\varphi;x,y)=x+U_{m,n}(x,y);\varphi(s,t)=s$

iii) $L_{m,n}(\psi;x,y)=y+V_{m,n}(x,y);\psi(s,t)=t$

iv) $L_{m,n}(\varphi^2+\psi^2;x,y)=x^2+y^2+W_{m,n}(x,y)$

v) $\{U_{m,n}(x,y)\}_{m,n\in\mathbb{N}}$, $\{V_{m,n}(x,y)\}_{m,n\in\mathbb{N}}$, $\{W_{m,n}(x,y)\}_{m,n\in\mathbb{N}}$

converge uniformly to zero.

Then the row $\{U_{m,n}f\}_{m,n\in\mathbb{N}}$ *converge to f uniform on D.*

The evaluation of the approximation order of a B-continuous B-continuity module, specified in:

1.2. Definition [1]: $B\subset D=[0,a]x[0,b]$ and $F: D \to \mathbb{R}$ a B-bounded function. The function $\omega_B: \mathbb{R}_+x\mathbb{R}_+ \to \mathbb{R}_+$ definited by:

$$\omega_B(\delta_1,\delta_2) = \sup_{(x,y),(x',y')\in D}\{|\Delta_2 f(x,y;x',y')| \mid |x-x'|\le\delta_1, |y-y'|\le\delta_2\}$$

is named the B-continuity module wich is associated to f.

Among the properties of the B-continuity module studied detailed in [1], in this work we will use that expressed in:

1.3. Theorem: *If* $\lambda_1, \lambda_2, \delta_1, \delta_2\in\mathbb{R}_+$ *we have the inequality:*

$$\omega_B(\lambda_1\delta_1,\lambda_2\delta_2) \le (1+\lambda_1)(1+\lambda_2)\,\omega_B(\delta_1,\delta_2) \quad .$$

2. The approximation of a B-continuous function by an operator of Bernstein type

In [3], B.Bleiman, P.L.Butzer and L.Hahn have introduced an linear and positive operator L_m, associated to a function f, wich is continuous an $[0,+\infty)$, being defined by:

$$(2.1) \qquad (L_m f)(x) = \frac{1}{(1+x)^m}\sum_{k=0}^{m}\binom{m}{n}x^k f\left(\frac{k}{m-k+1}\right),\ m\in\mathbb{N}.$$

A.Ciupa, in [4], has spread the 2.1 operator at two variables, associating to a function f continuous on $[0,+\infty)x[0,+\infty)$ the linear and positive operator $L_{m,n}$ defined by:

$$(2.2) \qquad (L_{m,n}f)(x,y) = \frac{1}{(1+x)^m(y)^n}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}x^k y^l f\left(\frac{k}{m-k+1},\frac{l}{n-l+1}\right) .$$

In the same work [4] is proved that if $D = [0,a]x[0,b]$ ($a>0$, $b>0$) and $f\in C(D)$, the row of general term (2.2) converge uniformly to f. Is given then an evaluation of the approximation order of the function $f\in C(D)$ by the row $(L_{m,n}f)_{m,n\in N}$.

The purpose of this work is to build an $U_{m,n}$ operators row with the property that the row $(U_{m,n}f_{m,n\in N}$ converge uniformly to $f\in C_b(D)$ and to estimate the approximation order of $f\in C_b(D)$ by the row $(U_{m,n}f)$.

We noote by L_m^x, L_n^y the parametrical extensious of the 1.1 operator defined by:

(2.3) $$(L_m^x f)(x,y) = \frac{1}{(1+x)^m}\sum_{k=0}^{m}\binom{m}{k}x^k f\left(\frac{k}{m-k+1},y\right)$$

(2.4) $$(L_n^y f)(x,y) = \frac{1}{(1+y)^n}\sum_{l=0}^{n}\binom{n}{l}y^l f\left(x,\frac{l}{n-l+1}\right)$$

By means of L_m^x, L_n^y operators we build the operator $U_{m,n}=$ $=L_m^x\oplus L_n^y=L_m^x+L_n^y-L_m^xL_n^y$ defined by:

(2.5)

$$(U_{m,n}f)(x,y) =$$
$$=\frac{1}{(1+x)^m(1+y)^n}\sum_{\substack{m,n\\k=0\\l=0}}\binom{m}{n}\binom{n}{l}x^k y^l\left[f\left(\frac{k}{m-k+1},y\right)+f\left(x,\frac{l}{n-l+1}\right)-f\left(\frac{k}{m-k+1},\frac{l}{n-l+1}\right)\right]$$

Because to $L_{m,n}$'s linearity the $U_{m,n}$ operator can be represented in the equivalent form:

(2.6) $$(U_{m,n}f)(x,y)=L_{m,n}[f(\cdot,y)+f(x,*)-f(\cdot,*);x,y]$$

In [4] are demonstred the $L_{m,n}$'s properties expressed in:

2.1. Theorem *The following affirmations are true:*

i) $L_{m,n}(1;x,y)=1$

ii) $L_{m,n}(t;x,y)=x-x\left(\frac{x}{1+x}\right)^m$

iii) $L_{m,n}(\tau;x,y)=y-y\left(\frac{y}{1+y}\right)^n$

iv) $L_{m,n}(t^2+\tau^2;x,y) \le x^2+y^2+\frac{2x(1+x)^2}{m+2}+\frac{2y(1+y)^2}{n+2}$, *for* $n\ge N(x)$, $m\ge N(y)$ *where* $N(x)=24(1+x)$, $N(y)=24(1+y)$.

As a result of 1.1. and 2.1 theorems we obtaine:

2.2. Theorem: *If* $f\in C_b(D)$, *the* $\{U_{m,n}f\}_{\substack{m\in N\\ n\in N}}$ *row of general therm (2.5) is converge to f uniform on D.*

We show from now on the evaluation of the order of the approximation of $f\in C_b(D)$ *order, by the row* $\{U_{m,n}f$.

A result used with this purpose is contained in:

2.3. Lemma [3], [4]: *If* $x\ge 0$, $m\ge 4x$ *we have the inequality:*

$$(2.7)\qquad x\left(\frac{x}{1+x}\right)^m \le \frac{2x(1+x)}{m+2}$$

2.4. Theorem: *If* $f\in C_b(D)$ *we have the assessment:*

$$(2.8)\quad |f(x,y)-(U_{m,n}f)(x,y)| \le (2+a)(2+b)\,\omega_B\left(\sqrt{\frac{2a(1+2a)}{m+2}},\sqrt{\frac{2b(1+2b)}{n+2}}\right)$$

Proof: Since $U_{m,n}(1;x,y)=1$, we have:

$$|f(x,y)-(U_{m,n}f)(x,y)| =$$

$$=\frac{1}{(1+x)^m(1+y)^n}\left|\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{n}\binom{n}{l}x^k y^l \Delta_2 f\left(x,y;\frac{k}{m-k+1},\frac{l}{m-k+1}\right)\right| \le$$

$$\le\frac{1}{(1+x)^m(1+y)^n}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}x^k y^l \omega_B\left(\left|x-\frac{k}{m-k+1}\right|,\left|y-\frac{l}{n-l+1}\right|\right).$$

Be $\delta_1>0$, $\delta_2>0$. We apply the 1.3 theorem with $\lambda_1=\frac{\left|x-\frac{k}{m-k+1}\right|}{\delta_1}$,

$\lambda_2=\frac{\left|y-\frac{l}{n-l+1}\right|}{\delta_2}$ and we have:

$$\omega_B\left(\left|x-\frac{k}{m-k+1}\right|,\left|y-\frac{l}{n-l+1}\right|\right)\le\left(1+\frac{\left|x-\frac{k}{m-k+1}\right|}{\delta_1}\right)\left(1+\frac{\left|y-\frac{l}{n-l+1}\right|}{\delta_2}\right)\omega_B(\delta_1,\delta_2)\quad .$$

Considering the precedent inequality, we obtain:

(2.8) $$|f(x,y)-(U_{m,n}f)(x,y)|\le$$

$$\le\omega_B(\delta_1,\delta_2)\left(1+\frac{1}{\delta_1}\cdot\frac{1}{(1+x)^m}\sum_{k=0}^{m}\left|x-\frac{k}{m-k+1}\right|\binom{m}{k}x^k\right)\cdot$$

$$\cdot\left(1+\frac{1}{\delta_2}\,\frac{1}{(1+y)^n}\sum_{l=0}^{n}\left|y-\frac{l}{n-l+1}\right|\binom{n}{l}y^l\right)\ .$$

Using the Schwarz inequality we have:

(2.9) $$\frac{1}{(1+x)^m}\sum_{k=0}^{m}\left|x-\frac{k}{m-k+1}\right|\binom{m}{k}x^k\le\sqrt{\frac{1}{(1+x)^m}\sum_{k=0}^{m}\left|x-\frac{k}{m-k+1}\right|^2\binom{m}{k}x^k}\ .$$

Applying 2.1 theorem (the ii, iv affirmations) and 2.3 lemma we obtain:

(2.10) $$\frac{1}{(1+x)^m}\sum_{k=0}^{m}\left|x-\frac{k}{m-k+1}\right|^2\binom{m}{k}x^k\le\frac{2x(1+x)^2(1+2x)}{m+2}\ .$$

Considering that $x\in[0,a]$, from (2.9) and (2.10) we have:

(2.11) $$\frac{1}{(1+x)^m}\sum_{k=0}^{m}\left|x-\frac{k}{m-k+1}\right|\binom{m}{k}x^k\le(a+1)\sqrt{\frac{2a(1+2a)}{m+2}}$$

By analogy we have:

(2.12) $$\frac{1}{(1+y)^n}\sum_{l=0}^{n}\left|y-\frac{l}{n-l+1}\right|\binom{n}{l}y^l\le(b+1)\sqrt{\frac{2b(1+2b)}{n+2}}\ .$$

The 2.8, 2.11, 2.12 inequalities lead to:

(2.13) $$|f(x,y)-(U_{m,n}f)(x,y)| \le$$

$$\le \omega_B(\delta_1,\delta_2)\left(1+\frac{a+1}{\delta_1}\sqrt{\frac{2a(1+2a)}{m+2}}\right)\left(1+\frac{b+1}{\delta_2}\sqrt{\frac{2b(1+2b)}{n+2}}\right)$$

We choose:

$$\delta_1=\sqrt{\frac{2a(1+2a)}{m+2}}\ ,\ \delta_2=\sqrt{\frac{2b(1+2b)}{n+2}}$$

and we obtain:

(2.14) $$|f(x,y)-(U_{m,n}f)(x,y)| \le (2+a)(2+b)\,\omega_B\left(\sqrt{\frac{2a(1+2a)}{m+2}}\ ,\ \sqrt{\frac{2b(1+2b)}{n+2}}\right)$$

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Received: March 9, 1994

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