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A FIXED POINT THEOREM FOR MAPPING WITH CONTRACTING ORBITAL DIAMETERS

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The well known contraction mapping principle has been extended in many directions until now. One of most interesting of them consists in taking a generalized contraction condition

$$d(Tx,Ty)\le a\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\},$$

instead the classical contraction condition

$$d(Tx,Ty)\le a\cdot d(x,y),$$

for a mapping $T:X\to X$, considered on a metric space (X,d), see [9]-[11], [12],[13],[14],[16] and especially [15].

The aim of this paper is to show that all these results can be reunied in a single one, using concepts as comparison function and generalized φ-contraction.

1. INTRODUCTION.

We need some definitions, examples and results from [1]-[8].

Definition 1. A map $\varphi:\mathbb{R}_+^5\to\mathbb{R}_+$ is called *comparison function* if

(i) $\varphi(u)\le\varphi(v)$, *for each* $u,v\in\mathbb{R}_+^5$, $u\le v$;

(ii) the sequence $(\psi^n(t))_{n\in\mathbb{N}}$ converges to zero, as $n\to\infty$, for each $t\in\mathbb{R}_+$, when $\psi:\mathbb{R}_+\to\mathbb{R}_+$ is given by

$$\psi(t)=\varphi(t,t,t,t,t),\ \textit{for each}\ t\in\mathbb{R}_+,\tag{1}$$

(on $\mathbb{R}^5$ we consider the partial order relation).

The following are examples of comparison functions (called 5-comparison functions in [8]).

Examples.

1° $\varphi:\mathbb{R}_+^5\to\mathbb{R}_+$, given by

$$\varphi(t)=a\cdot\max\{t_1,t_2,t_3,t_4,t_5\}\ ,\ \textit{for each}\ t=(t_1,t_2,t_3,t_4,t_5)\in\mathbb{R}_+^5\ ,$$

where $a\in(0,1)$ is a constant.

2° $\varphi:\mathbb{R}_+^5\to\mathbb{R}_+$, given by

$$\varphi(t)=a\cdot\max\{t_1,t_2,t_3,t_4,\frac{t_4+t_5}{2}\}\ ,\ t=(t_1,t_2,t_3,t_4,t_5)\in\mathbb{R}_+^5\ ,$$

if $a\in(0,1)$.

3° For $a\in\left[0,\frac{1}{2}\right]$, $\varphi:\mathbb{R}_+^5\to\mathbb{R}_+$, given by

$$\varphi(t)=at_2+at_3\ ,\ \textit{for each}\ t=(t_1,t_2,t_3,t_4,t_5)\in\mathbb{R}_+^5\ .$$

4° For $a,b\in\mathbb{R}_+$, $a+2b<1$, $\varphi:\mathbb{R}_+^5\to\mathbb{R}_+$,

$$\varphi(t)=at_1+b(t_2+t_3)\ ,\ t=(t_1,t_2,t_3,t_4,t_5)\in\mathbb{R}_+^5\ .$$

5° For $a\in[0,1]$, $\varphi:\mathbb{R}_+^5\to\mathbb{R}_+$,

$$\varphi(t)=a\cdot\max\{t_2,t_3\}\ ,\ t=(t_1,t_2,t_3,t_4,t_5)\in\mathbb{R}_+^5\ .$$

6° Let $a,b,c\in\mathbb{R}_+$, $a<1$, $b<\frac{1}{2}$ and $c<\frac{1}{2}$. If, for a certain $t\in\mathbb{R}_+$, $\varphi(t)$ is given by one of the following values

$$at_1\ ,\ b(t_2+t_3)\ \textit{or}\ c(t_4+t_5)\ ,$$

then the obtained function $\varphi:\mathbb{R}_+^5\to\mathbb{R}_+$ is a comparison function.

2. GENERALIZED CONTRACTIONS.

For (X,d) a metric space and $T: X\to X$ an operator, we denote

$F_T:=\{x\in X/Tx=x\}$,

$\left(T^{n}_{x_0}\right)_{n\in N}$, the sequence of successive approximations corresponding to the initial approximation x_0;

$O(x;T):=\{x, Tx, T^2x, \ldots\}$;

$\delta(A)=sup\{d(x,y)/x,y\in A\}$, $A\subset X$.

The following lemmas are generalizations of Lemma 1 and Lemma 2 from CIRIC,L.[9].

LEMMA 1. Let (X,d) be a metric space and $T:X\to X$ a *generalized φ-contraction*, i.e. an operator for which there exists a comparison function $\varphi:\mathbb{R}^5_+\to\mathbb{R}_+$, such that

$$d(Tx,Ty)\le\varphi(d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)),\ x,y\in X. \quad (2)$$

Then, for $n\in N$ and any $x\in X$, if $i,j\in\{1,2,\ldots,n\}$, we have

$$d\left(T^i_x, T^j_x\right)\le\psi(\delta[O(x;n)]),$$

where $\psi:\mathbb{R}_+\to\mathbb{R}_+$ is given by (1).

Proof. Let $x_0\in X$ be arbitrary taken, and $x_n=T^n x_0$, $n\ge 0$, the sequence of succesive approximations.
From $i,j\in\{1,2,\ldots,n\}$ it results $\{i-1,j-1,i,j\}\subset\{0,1,2,\ldots,n\}$, hence

$$x_{i-1}, x_i, x_{j-1}, x_j\in O(x;n).$$

Then, from the contraction condition (2), we have

$$d(x_i,x_j)=d(Tx_{j-1},Tx_{j-1})\le$$

$$\le\varphi(d(x_{i-1},x_{j-1}), d(x_{i-1},x_j), d(x_{j-1},x_j), d(x_{i-1},x_j), d(x_{j-1},x_i)).$$

But

$$d(x_p,x_q)\le\delta[O(x,n)],\ \textit{for each}\ p,q\in\{i-1,j-1,i,j\},$$

and φ is monotone increasing, then

$$d(x_i,x_j)\le\varphi(r,r,r,r,r)=\psi(r),$$

where we have denoted

$$r=\delta[0(x;n)] .$$

The proof is complete.

Remark 1. For each $n\in\mathbb{N}^*$, there is $k\leq n$ so that

$$d(x,T^kx)=\delta[0(x;n)] ,$$

since

$$\psi(r)\leq r \text{ , for each } r\geq 0 \text{ (see[1]-[7]) .}$$

LEMMA 2. If $T: X\to X$ is a φ-contraction and, in addition, φ is such that the function $h: \mathbb{R}_+ \to \mathbb{R}_+$,

$$h(t)=t-\psi(t)=t-\varphi(t,t,t,t,t) \text{ , } t\in\mathbb{R}_+ \text{ ,} \tag{3}$$

is a bijection, then, for any $n\in\mathbb{N}$, we have

$$\delta[0(x;n)]\leq h^{-1}(d(x,Tx)) \text{ , } \forall x\in X .$$

Proof. Let $n\in\mathbb{N}$ be arbitrary taken. From remark 1 it results that there exists $k\leq n$, such that

$$d(x,T^kx)=\delta[0(x;n)] ,$$

hence, applying lemma 1, we obtain

$$\delta[0(x;n)]=d(x,T^kx)\leq$$

$$\leq d(x,Tx)+d(Tx,T^kx)\leq d(x,T(x))+\psi(\delta[0(x;n)]) ,$$

which lead to

$$\delta[0(x;n)]-\psi(\delta[0(x;n)])\leq d(x,Tx) \text{ , } x\in X, n\in\mathbb{N} .$$

But h is bijective and monotone increasing (because φ is also increasing), hence h^{-1} is increasing too, and the conclusion follows from the last inequality.

3. A FIXED POINT THEOREM FOR GENERALIZED CONTRACTIONS

The main result of this paper is given by the following

THEOREM 1. *Let (X,d) be a complete metric space and $T: X \to X$ a φ-contraction with φ such that the function ψ given by (1) is continuous and the function h given by (3) is a bijection.*
Then

(i) $F_T=\{x^*\}$;

(ii) The sequence of succesive approximations, $x_n=T^n x_0,\ n\geq 0$ *converges to* x^* , *for each* $x_0\in X$;

(iii) The following estimation holds

$$d(x_n,x^*)\leq \psi^n(h^{-1}(d(x_0,x_1))) .$$

Proof. Let $x_0\in X$ and $m,n\in N$, $n<m$.
We take $i=1, j=m-n+1, x=T^{n-1}x_0=x_{n-1}$ and apply Lemma 1.
It results

$$d(x_n,x_m)=d(Tx_{n-1},Tx_{m-1})\leq \psi(r_1) , \tag{4}$$

where

$$r_1=\delta[0(x_{n-1},m-n+1)] .$$

Now, from remark 1, there exists k_1, $1\leq k_1\leq m-n+1$, such that

$$\delta[0(x_{n-1},m-n+1)]=d(x_{n-1},T^{k_1}x_{n-1}) . \tag{5}$$

Using again lemma 1, we obtain

$$d(x_{n-1},T^{k_1}x_{n-1})=d(T(x_{n-2}),T^{k_1+1}(x_{n-2}))\leq \psi(r_2) , \tag{6}$$

where

$$r_2=\delta[0(x_{n-2},k_1+1)] .$$

But $k_1+1\leq m-n+2$ and ψ is monotone increasing, hence, from (4)-(6), we obtain

$$d(x_n, x_m) \le \psi^2(\delta[O(x_{n-2}, m-n+2)]) ,$$

and, by induction, we deduce

$$d(x_n, x_m) \le \psi^n(\delta[O(x_0; m)]) .$$

Now, using lemma 2, it results

$$d(x_n, x_m) \le \psi^n(r_3) , \tag{7}$$

where

$$r_3 = h^{-1}(d(x_0, x_1)) .$$

But φ is comparison function, hence

$$\psi^n(r) \to 0 \ , \ as\ n \to \infty \ , \ for\ each\ r \in \mathbb{R}_+ ,$$

which shows, together with (7), that (x_n) is a Cauchy sequence in the complete metric space (X,d). This means (x_n) is convergent. Let

$$x^* = \lim_{n\to\infty} x_n .$$

We shall show that x^* is a fixed point of T.
Indeed, for each $n \in \mathbb{N}$,

$$d(x^*, Tx^*) \le d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \le d(x^*, x_{n+1}) +$$

$$+ \varphi(s(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), d(x_n, Tx^*), d(x_{n+1}, x^*)) . \tag{8}$$

If

$$\max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), d(x_n, Tx^*), d(x_{n+1}, x^*)\} = d(x^*, Tx^*) ,$$

then using the monotonicity of φ, from (8) we obtain

$$d(x^*, Tx^*) \le d(x^*, x_{n+1}) + \psi(d(x^*, Tx^*)) ,$$

that is

$$d(x^*, Tx^*) \le h^{-1}(d(x^*, x_{n-1})) . \tag{9}$$

But h^{-1} is monotone increasing, positive and $h^{-1}(0)=0$, hence h^{-1} is continuous at zero. Taking $n \to \infty$ in (9), we obtain

$$d(x^*,Tx^*)=0 ,$$

wich means $x^*\in F_T$.

If

$$\max\{d(x_n,x^*),d(x_n,x_{n+1}),d(x^*,Tx^*),d(x_n,Tx^*),d(x_{n+1},x^*)\}=d(x_n,x^*) ,$$

then from (8) we obtain

$$d(x^*,Tx^*)\le d(x_n,x^*)+\psi(d(x_n,x^*)) ,$$

which yields, in view with the continuity of ψ at 0, and taking $n \to \infty$,

$$d(x^*,Tx^*)\le 0 ,$$

hence $d(x^*,Tx^*)=0$, that is $x^*\in F_T$. Let's remark that if the maximum is $d(x_{n+1},x^*)$ or $d(x_n,x_{n+1})$, the proof is similar to the previous case. If, finally,

$$\max\{d(x_n,x^*),d(x_n,x_{n+1}),d(x^*,Tx^*),d(x_n,Tx^*),d(x_{n-1},x^*)\}=d(x_n,Tx^*) ,$$

then, from (8) it results

$$d(x^*,Tx^*)\le d(x_n,x^*)+\psi(d(x_n,Tx^*)) .$$

Taking $n \to \infty$ and, using the continuity of ψ, we obtain

$$d(x^*,Tx^*)-\psi(d(x^*,Tx^*))\le 0 ,$$

that is

$$h^{-1}(d(x^*,Tx^*))\le 0 ,$$

which means

$$h^{-1}(d(x^*,Tx^*))=0 \ i.e.\ \ d(x^*,Tx^*)=0 .$$

The unicity of the fixed point is proved as follows.

Let $x_1^*, x_2^* \in F_T, x_1^* \neq x_2^*$. This means $d(x_1^*, x_2^*) > 0$ and then

$$d(x_1^*, x_2^*) = d(T^n x_1^*, T^n x_2^*) \le \psi^n(\delta[O(x_1^*; m)]) = \psi^n(\delta(\{x_1^*\})) = \psi^n(0) = 0,$$

contradiction.

Now, (i) and (ii) are proved. In order to obtain (iii), we take $n \to \infty$ in (7).

The proof is now complete.

Remark.

1) If φ is as in example 1°, from our theorem we obtain a result from CIRIC,L. [9];

2) If φ is as in example 3°, from theorem 1 we obtain the wellknown Kannan's theorem, KANNAN,R. [11].

THEOREM 2. *Let (X,d) be a complete metric space and $T: X \to X$ an operator for which there exists $a \in \left[0, \frac{1}{2}\right)$ such that*

$$d(Tx, Ty) \le a\,[d(x, Tx) + d(y, Ty)] \quad , \forall x, y \in X .$$

Then

$$F_T = \{x^*\} .$$

Remarks.

1) Theorem 1 gives the same error estimates as in theorem 3.1.1 from RUS,A.I. [13], but in the particular case of the Kannan's theorem, our error estimates is slower than the one given by theorem 3.2.1, from RUS,A.I. [12];

2) If φ is as in example 4°, then from theorem 1 we obtain a fixed point theorem given by REICH,S.(1971) and RUS,A.I.(1971), see TASCOVIĆ,M. [15];

3) If φ is as in example 5°, then from theorem 1 we obtain a fixed point theorem given BIANCHINI,M.(1972) and DUGUNDJI,J.(1976);

4) If φ is as in example 6°, then from theorem 1 we obtain a very interesting fixed point theorem established by ZAMFIRESCU,T. [16];

5) By considering for φ other particular expressions, theorem 1 furnishes various fixed point theorems established by: Sehgal,V.

(1972), Roades,B.E.(1977) and Chatterjea,S.(1972), Hardy,G.E. and Rogers,T.D.(1973), Iseki,K.(1975), Kurepa,S.(1976), Ćirić,L.(1971) and many others, see TASKOVIĆ,M.[15] and RUS,A.I. [14];

6) Theorem 1 in our paper extracts the unifying principle for all these fixed point theorems, that is

$$\psi(t)=\alpha t\ ,\ 0\leq\alpha<1,$$

whith α adequate chosen, for any comparison function φ satisfying the conditions of theorem 1.

For example, for φ as in example 3, $\alpha = 2a<1$, for φ as in example 4, $\alpha = a+2b<1$ for φ as in example 5, $\alpha = a$, and for φ as in example 6, $\alpha = \min(a,b,c)$;

7) If $T: X \to X$ is a contraction then T is continuous, but if T is a generalized φ-contraction, T is generally discontinuous, see ĆIRIĆ,L.[9] or RUS,A.I.[14];

8) A similar result to theorem 1 in this paper is given in RUS,A.I.[14],theorem 1, where ψ satisfies a weaker condition but φ is claimed to satisfy stronger conditions then those in our paper.

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