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ON THE COMMON ORIGINE OF SOME ITERATIVE METHODS

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1. Introduction. The aim of the paper is to analyse the common origine of some linear iterative methods applied to solving the linear systems of the form

$$Ax=b \ , \tag{1.1}$$

where A is a nxn real matrix, b is a column n vector and x is a column n unknown vector.

All the methods presented here can be considered having a common source, namely some generalization of Euclid's algorithm for the common divisor of two integers, and a convenient choice of the coefficients involved. Thus we will derive many well-known linear iterative methods of first order and of second order for solving the system (1.1), see [3], [4], [5].

This idea is successfully used by E.A.Lipitakis in [1].

If a, b are integers, a>b then Euclid's algorithm is given by the relations

$$r_i = r_{i+1} q_{i+1} + r_{i+2} \ ; i=-1,0,1,2\ldots, \tag{1.2}$$

where $r_{-1}=a$, $\gamma_0=b$.

We know that Euclid's algorithm (1.2) has a finite number of relations, that is there is $n\in N$ so that $r_{n+1}=0$, and then r_n is the largest common divisor of the numbers a and b.

2. The generalization of the Euclid's algorithm.

Let the recurrence relation

$$r_{k+1} = \alpha_k r_k + \beta_k r_{k-1} + \gamma_k \ , \ k\geq 0 \ , \tag{2.1}$$

be, where (α_k), (β_k), (γ_k) are given sequence of real numbers. It is easy to see that Euclid's algorithm (1.2) can be obtained by a particular choice of the coefficients α_k, β_k, γ_k from (2.1). Indeed, if we take k=i+1, i=-1,0,1,2,... in (2.1), we have

$$r_{i+2} = \alpha_{i+1} r_{i+1} + \beta_{i+1} r_i + \gamma_{i+1} \,, \tag{2.2}$$

and if we choose, for i≥-1,

$$\alpha_{i+1} = -q_{i+1} \,,\ \beta_{i+1} = 1 \,,\ \gamma_{i+1} = 0 \,,$$

then we find Euclid's algorithm (1.2).

Let's now consider the recurrent relation given by (2.1) as an independent relation.
This recurrent relation, for given (α_k), (β_k), (γ_k), means a general linear recurrent relation of the second order and this recurrent relation suggests the form of the linear iterative methods of first and of second order for solving systems of the form (1.1).

Remark 2.1. If in (2.1) α_k, β_k, γ_k are constants then we can get the general term of the sequence (r_k) which satisfies this recurrent relation.

3. Derive some linear iterations of first order.

Suppose that for the numerical solving of the system (1.1) we use the following iterative method suggested by the relation (2.1).

$$x^{(k+1)} = \alpha_k x^{(k)} + \beta_k x^{(k-1)} + \gamma_k \,,\ k \geq 0 \,, \tag{3.1}$$

where $x^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)^T$ is a column vector giving the k-th approximation of the solution of the system (1.1), (α_k), (β_k) are given sequence of real numbers or may be given sequence of real nxn matrices and (γ_k) is a given n column vector.

Next, we will choose α_k, β_k, γ_k.
First, we observe that if

$$A = (a_{ij}) \,,$$

then we can have for A the following decomposition

$$A = L + D + U \,, \tag{3.2}$$

with $D=\mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$ and L is a strictly lower triangular (made by the elements under the diagonal of A, completed by zeros) and U is a strictly upper triangular (made by the elements over the diagonal of A, completed by zeros).

We assume that the matrix A has $a_{ii}>0$, $i=1,2,\ldots,n$, because if there is a $a_{kk}<0$ it is possible to change the order of equations of the system (1.1) or to change the order of unknowns $x_1, x_2, \ldots, x_n$ so as to have all $a_{ii}>0$.

a) We have, first, the following choice for the coefficients α_k, β_k, γ_k from (3.1)

$$\alpha_k = -D^{-1}(L+U)\ ,\ \beta_k=0\ ,\ \gamma_k=D^{-1}b\,. \tag{3.3}$$

Then we obtain the linear iterative method of order first

$$x^{(k+1)} = -D^{-1}(L+U)\,x^{(k)} + D^{-1}b,\ k=0,1,2,\ldots$$

known as the **iterative method of Jacobi**, which is a stationary method.

Remark 3.1. It is known (see for example [4], pp.431) that the method (3.4) converges to the solution of the sysstem (1.1) if

$$\|D^{-1}(L+U)\|<1\ , \tag{3.5}$$

where the norm is the euclidian norm.

b) If we have in (3.1) the choice

$$\alpha_k = -(D+L)^{-1}\cdot U,\ \beta_k=0\ ,\ \gamma_k=(D+L)^{-1}\cdot b, \tag{3.6}$$

in the assumption (3.2), then we obtain a new linear iterative method of first order

$$x^{(k+1)} = -(D+L)^{-1}Ux^{(k)} + (D+L)^{-1}b, \tag{3.7}$$

known as the **Gauss-Seidel method** for solving the system (1.1).

Remark 3.2. A sufficient condition for the convergence of the method (3.7) is

$$\|(D+L)^{-1}U\|<1\ .$$

Remark 3.3. If the matrix A is positively defined, then, (see [4], pp.432) the method of Gauss-Seidel (3.7) converges independent of the choice of the initial vector $x^{(0)}$ and this method has a larger speed of convergence than the Jacobi method.

c) For a new choice of coefficients in (3.1) we use the following representation of the matix A

$$A=\frac{1}{\mu}(D+\mu L)+\frac{1}{\mu}[(\mu-1)D+\mu U], \tag{3.8}$$

with $\mu\neq 0$ and the assumption that the matrix $\frac{1}{\mu}(D+\mu L)$ is nonsingular.

We observe that (3.8) is another form of the decomposition (3.2) of the matrix A.

Now we take for coefficients α_k, β_k, γ_k from (3.1)

$$\alpha_k=-\left[\frac{1}{\mu}(D+\mu L)\right]^{-1}\cdot\left[\frac{1}{\mu}(\mu-1)D+\mu U\right]$$
$$\beta_k=0\ ,\ \gamma_k=\left[\frac{1}{\mu}(D+\mu L)\right]^{-1}\cdot b, \tag{3.9}$$

and so we obtain a one parameter class of linear iterative method of the first order

$$x^{(k+1)}=-\left[\frac{1}{\mu}(D+\mu L)\right]^{-1}\left[\frac{1}{\mu}(\mu-1)D+\mu U\right]x^{(k)}+\left[\frac{1}{\mu}(D+\mu L)\right]^{-1}\cdot b\ . \tag{3.10}$$

Remark 3.4. If μ=1 in (3.10) the method (3.10) come to Gauss-Seidel method (3.7).

The method given (3.10) is named the **method of successive relaxation** and if $0<\mu<1$ is named **the method of underrelaxation** and if $1<\mu<2$ is named the **method of overrelaxation.**

We mention here a convergence result on the method given by (3.10).

Theorem 3.1. (Marinescu Gh., [2]). *The method given by (3.10) converges to the solution x^*, of the system (1.1), if and only if $0<\mu<2$, and this convergence is characterized by*

$$\|x^{(k)}-x^*\|_A \leq \frac{q^n}{1-q}\|x^{(1)}-x^{(0)}\|_A , \tag{3.11}$$

where

$$q=\left\|\left[\frac{1}{\mu}(D+\mu L)\right]^{-1}\left[\frac{1}{\mu}((\mu-1)D+\mu U)\right]\right\|_A<1 ,$$

and the norm is defined by the scalar product

$$\langle x,y\rangle_A=\langle Ax,y\rangle .$$

d) We make now an other choice for the coefficients in (3.1). For this, let M be a square matrix of order n so that its inverse M^{-1} can be computed in a simple way.
Then we choose in (3.1)

$$\alpha_k=I-u_kM^{-1}\cdot A,\ \beta_k=0,\ \gamma_k=u_kM^{-1}b, \tag{3.12}$$

(u_k) being a given sequence and I the unit matrix of order n.
Thus we derived the following linear nonstationary method of first order

$$x^{(k+1)}=(I-u_kM^{-1}A)x^{(k)}+u_kM^{-1}b \tag{3.13}$$

called in literature as the **Richardson accelerating method.**

Remark 3.5. For u_k=u=constant, we get a linear stationary method called the **overrelaxation method of Jacobi** and if u_k=u and M=D, D as in (3.2) then we obtain the method

$$x^{(k+1)}=(I-uD^{-1}A)x^{(x)}+uD^{-1}b , \tag{3.14}$$

called the **method of simultaneous relaxation.**
We recall a convergence result about the method (3.14), without demonstration.

Theorem 3.2. (R.Varga, [5]). *The iterative method (3.14) applied for solving (1.1) converges to the solution of the system (1.1) for arbitrary initial vector $x^{(0)}$, if and only if $0<u<\frac{2}{\lambda}$, where λ represents the largest positive eigenvalue of the matrix $D^{-1}A$.*

If in (3.1) we choose the coefficients with $\beta_k \neq 0$, then we can obtain various linear methods of second order like Cebîşev type methods, conjugate gradient methods etc.

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