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GENERALIZED BLENDING OPERATORS OF FAVARD-SZASZ TYPE

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SUMMARY: A Favard Szasz blending operator is built; the approximation of a B-continuous function with two variables is studied using this type of operators and an estimate of the order of approximation is givin.

1. **PRELIMINARIES:** Let be $g(z)=\sum_{n=0}^{\infty} a_n z^n$ an analytical function on the disk $|z|<R,\ R>1$, where $g(1)\neq 0$.
The Appell polynomials $P_k(x)$, $k\geq 0$ are defined in [1] by:

$$g(u)\,e^{ux}=\sum_{k=0}^{\infty} P_k(x)\,u^k \tag{1.1}$$

We associate to each function $f:[0,+\infty)\to\mathbb{R}$ the row of linear operators wich are defineted by:

$$(P_n f)(x)=\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} P_k(nx)\,f\left(\frac{k}{n}\right) \tag{1.2}$$

wich in yhe case $g(z)\equiv 1$ is reduced at the classic Favard-Szasz operators studied in [5], [6].

Let's suppose now that $g(z)=\sum_{n=0}^{\infty} a_n z^n,\ h(z)=\sum_{n=0}^{\infty} b_n z^n$ are analytical functions in the disk $|z|<R$, $R>1$ and let's note by P_k, q_k, k=0,1,... the Appell polynomials generated by g and h, respectivly by the relations:

(1.3)
$$g(u)\,e^{ux}\equiv\sum_{k=0}^{\infty}P_k(x)\,u^k,\ h(v)\,e^{vx}=\sum_{k=0}^{\infty}q_n(x)\,v^k$$

Be f:[0,+∞)x[0,+∞) → R and let's note by xP_m, $y\overline{P_n}$ the parametrical extensions of the operator (1.2), given respectively by:

(1.4)
$$(xP_m f)\,(x,y)=\frac{e^{-mx}}{g(1)}\sum_{k=0}^{\infty}P_k(mx)\,f\left(\frac{k}{m},y\right)$$

(1.5)
$$(y\overline{P}_n f)\,(x,y)=\frac{e^{nx}}{h(1)}\sum_{j=0}^{\infty}q_j(jy)\,f\left(x,\frac{j}{n}\right)$$

A.Ciupa [4] associate to a function f:[0,+∞)x[0,+∞) → ℝ the generalized operator of Favard-Szasz type wich is defineted by:

(1.6)
$$(P_{m,n}f)\,(x,y)=\frac{e^{-mx}\cdot e^{-ny}}{g(1)\cdot h(1)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}P_k(mk)\,q_j(ny)\,f\left(\frac{k}{m},\frac{j}{n}\right)$$

Let's note that $P_{m,n}$ operator wich interfere in (1.6) is the tensorial product of the parametrical extensions xP_m, $y\overline{P}_n$.

2. GENERALIZED FAVARD-SZASZ BLENDING OPERATORS.

We denote $\mathscr{E}_2$ the space of the functions f:[0,+∞)x[0,+∞) → ℝ wich are of exponential type:

(2.1)
$$\mathscr{E}_2=\{f:[0,+\infty)\,x[0,+\infty)\to\mathbb{R}\,|\,(\exists)\,A,B\in\mathbb{R}\ so\ that\ |f(x,y)|\le e^{Ax+By}\}$$

To each function $f\in\mathscr{E}_2$ we associate the parametrical extensions of the generalized Favard-Szasz operators, defineted at (1.4), respectively (1.5).

The results of 2.1 lemma are evident:

2.1. Lemma:

i) The operators xP_m, $y\overline{P_n}$ defineted at (1.4), (1.5) are linear

ii) If $\frac{a_n}{g(1)}\ge 0,\ \frac{b_n}{g(1)}\ge 0\ \ (\forall)\,n\in\mathbb{N}$ *, then the operators xP_m, yP_n are positive.*

2.2. Lemma: *If* $\frac{a_n}{g(1)} \geq 0$, $\frac{b_n}{g(1)} \geq 0$, $(\forall) n \in \mathbb{N}$ *, then the rows of positive linear operators* $({}_xP_m)_{m\in\mathbb{N}}$, $({}_y\overline{P}_n)_{n\in\mathbb{N}}$ *have the properties:*

$$xP_m e_i \to e_i,\ y\overline{P}_n e_i \to e_i\ (i=0,1,2)$$

uniformely on [0,1] when $m,n \to \infty$ $(e_i(x)=x^i,\ i=0,1,2)$.

Proof: From [5], the following equalities are known:

(2.2) $$(xP_m e_0)(x)=1$$

(2.3) $$(xP_m e_1)(x)=x+\frac{1}{m}\cdot\frac{g'(1)}{g(1)}$$

(2.4) $$(xP_m e_2)(x)=x^2+\frac{x}{m}\left(1+2\frac{g'(1)}{g(1)}\right)+\frac{1}{m^2}\frac{g''(1)+g'(1)}{g(1)}$$

from wich it result that $xP_m e_i \to e_i$ (i=0,1,2), uniformly on [0,1]. In a similar way we stablish that $y\overline{P}_n e_i \to e_i\ (i=0,1,2)$, uniformly on [0,1].

2.3. Definition: Let be $f \in \mathscr{E}_2$. The operator:

(2.5) $$U_{m,n}=xP_m+y\overline{P}_n-xP_m \circ y\overline{P}_n$$

is called blending operator of Favard-Szasz type.

2.4. Lemma *The approximating* $U_{m,n}f$ *of a function* $f \in \mathscr{E}_2$ *defined by the (2.5) operator admits the representation:*

(2.6) $$(U_{m,n}f)(x,y)=\frac{e^{-mx}e^{-ny}}{g(1)h(1)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}p_k(mx)\,q_j(ny)\left[f\left(\frac{k}{m},y\right)+f\left(x,\frac{j}{n}\right)-f\left(\frac{k}{m},\frac{j}{n}\right)\right]$$

Proof: The equality result directly from (1.4), (1.5), (2.5) and from the properties of the unidimensional operators of Favard-Szasz type.

The B-continous function notion is that from [3]. As a consequence of the 2.2 lemma and of the 3 corolary from [3] we have:

2.5. Theorem *If $f\in\mathscr{E}_2$ is a B-continuous function on [0,1]x[0,1], then* $\lim\limits_{m,n\to\infty}(U_{m,n}f)(x,y)=f(x,y)$ *, uniformly on [0,1]x[0,1].*

3. THE ORDER OF THE APPROXIMATION OF A B-BOUNDED FUNCTION BY THE ROW $\{u_{m,n}f\}$

The bidimensional difference Δ_f, the B-modulus of continuity ω_B and the B-bounded function notion are those from [2].

3.1. Theorem *If f is B-bounded on [0,1]x[0,1] we have the estimation:*

$$|f(x,y)-(U_{m,n}f)(x,y)|\le$$

$$\le\left(1+\sqrt{1+\frac{g''(1)+g'(1)}{mg(1)}}\right)\left(1+\sqrt{1+\frac{h''(1)+h'(1)}{n-h(1)}}\right)\omega_B\left(\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right) \tag{3.1}$$

Proof: *Using (2.6) we obtain*

$$|f(x,y)-(U_{m,n}f)(x,y)|\le$$

$$\le\frac{e^{-mx}}{g(1)}\cdot\frac{e^{-ny}}{h(1)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}P_k(m,x)\,q_j(n,y)\,|\Delta_f(x,y;\frac{k}{m},\frac{j}{n})| \tag{3.2}$$

From the properties of the B modulus of continuity ω_B we obtain:

$$|\Delta_f\left(x,y;\frac{k}{m},\frac{j}{n}\right)|\le\omega_B\left(|x-\frac{k}{m}|,|y-\frac{j}{n}|\right)=\omega_B\left(\frac{1}{\delta_1}|x-\frac{k}{m}|\delta_1,\frac{1}{\delta_2}|y-\frac{j}{n}|\delta_2\right)\le$$

$$\le\left(1+\frac{1}{\delta_1}|x-\frac{k}{m}|\right)\left(1+\frac{1}{\delta_2}|y-\frac{j}{n}|\right)\omega_B(\delta_1,\delta_2)$$

Coming back in (3.2) we obtain:

$$|f(x,y)-(U_{m,n}f)(x,y)|\le$$

$$\le\left(1+\frac{1}{\delta_1}\frac{e^{-mx}}{g(1)}\sum_{k=0}^{\infty}P_k(mx)\,|x-\frac{k}{m}|\right)\left(1+\frac{1}{\delta_2}\frac{e^{-ny}}{-h}\sum_{j=0}^{\infty}q_j(ny)\,|y-\frac{j}{n}|\right)\cdot\omega_B(\delta_1,\delta_2) \tag{3.3}$$

The Cauchy's inequality and the properties of the unidimensional operators of Favard Szasz type lead to:

$$\text{(3.4)}\quad \begin{aligned}&\sum_{k=0}^{\infty} P_k(mx)\,|x-\frac{k}{m}|\le\\ &\le\sqrt{\sum_{k=0}^{\infty} P_k(m,x)}\sqrt{\sum_{k=0}^{\infty} P_k(mx)\,(x-\frac{k}{m})^2}=g(1)\,e^{mx}\sqrt{\frac{x}{m}+\frac{1}{m^2}\,\frac{g''(1)+g'(1)}{g(1)}}\end{aligned}$$

$$\text{(3.5)}\quad \begin{aligned}&\sum_{j=0}^{\infty} q_j(ny)\,|y-\frac{j}{n}|\le\\ &\le\sqrt{\sum_{j=0}^{\infty} q_j(ny)}\sqrt{\sum_{j=0}^{\infty} q_j(ny)\,(y-\frac{j}{h})^2}=h(1e^{ny}\sqrt{\frac{y}{n}+\frac{1}{n^2}\cdot\frac{h''(1)+h'(1)}{h(1)}}\end{aligned}$$

From (3.3), (3.4), (3.5) we obtain:

$$\text{(3.6)}\quad \begin{aligned}&|f(x,y)-(U_{m,n}f)(x,y)|\le\\ &\le\left(1+\frac{1}{\delta_1}\sqrt{\frac{x}{m}+\frac{g''(1)+g'(1)}{m^2g(1)}}\right)\left(1+\frac{1}{\delta_2}\sqrt{y_n+\frac{h''(1)+h'(1)}{n^2}}\right)\cdot\omega_B(\delta_1\delta_2)\end{aligned}$$

For (x,y)∈[0,1]x[0,1] we obtain:

$$\text{(3.7)}\qquad \frac{x}{m}+\frac{g''(1)+g'(1)}{m^2g(1)}\le\frac{1}{m}+\frac{g''(1)+g'(1)}{m^2}$$

$$\text{(3.8)}\qquad \frac{y}{n}+\frac{h''(1)+h'(1)}{n^2g(1)}\le\frac{1}{n}+\frac{h''(1)+h'(1)}{n^2}$$

Choosing $\delta_1=\frac{1}{\sqrt{m}}$, $\delta_2=\frac{1}{\sqrt{n}}$ in (3.6) and taking account of (3.7), (3.8) we obtain (3.1).

4. THE PARTICULAR CASE OF FAVARD SZASZ BLENDING OPERATORS

In the particular case wich g(z)=1, from (1.2) we obtain the classic Favard Szasz operators definited by:

$$(S_m f)(x) = e^{-mx}\sum_{k=0}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{k}{m}\right) \tag{4.1}$$

The Favard Szasz blending operator associated to a funcţion $f \in \mathscr{E}$ has the expression:

$$(S_{m,n}f)(x,y) = e^{-(mx+ny)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(mk)^k}{k!}\frac{(ny)^j}{j!}\left[f\left(\frac{k}{m},y\right)+f\left(x,\frac{j}{n}\right)-f\left(\frac{k}{m},\frac{j}{n}\right)\right] \tag{4.2}$$

and the estimation similar to (3.1) is:

$$|f(x,y)-(S_{m,n}f)(x,y)| \le 4\omega_B\left(\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right) \tag{4.3}$$

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