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Asymptotic Properties of Solutions of Second Order Nonlinear Differential Equations with Deviating Argument

Vincent Šoltés

Abstract [1]

In the paper sufficient conditions are found under which for all oscillatory solutions of the equation

$$(r(t)y'(t))' + a(t)f(y(g(t))) = b(t),$$

there is $\lim_{t\to\infty} y(t) = 0$. *Sufficient conditions are also found so that all solutions of this equation are unbounded.*

1. Introduction

The second order differential equation of the form

$$(r(t)y'(t))' + a(t)y(g(t)) = f(t) \tag{1}$$

is investigated in the paper [4]. Sufficient, necessary and sufficient conditions are found under which all oscillatory solutions of the equation (1) approach zero asymptotically. Sufficient conditions are also found so that oscillatory solutions of (1) must be unbounded. Proofs of the Theorems 1 and 9 in the paper [4] are not correct entirely.

We will investigate differential equation of the form

$$(r(t)y'(t))' + a(t)f(y(g(t))) = b(t). \tag{2}$$

The aim of the paper is to complete and to generalize some results for the equation (2) as well as to remove incorrection of the Theorems 1 and 9 from the paper [4].

[1] Key words: Oscillatory, nonoscillatory, asymptotic, unbounded.

2. Definition and assumptions

It will be assumed for the rest of the paper that

(i) $r(t),\ a(t),\ b(t),\ g(t),\ f(t) : R \to R$ are continuous ; R is a real line,
(ii) $r(t) > 0,\ g(t) > 0,\ g(t) \to \infty$ as $t \to \infty,\ g(t) < t$,
(iii) $uf(u) > 0 \quad for \quad u \neq 0$.

The function $y(t) \in C[t_0, \infty)$ is called oscillatory if it has arbitrarily large zeros in $[t_0, \infty)$. Otherwise $y(t)$ is called nonoscillatory. In this paper we investigate only solution of (2) which are defined for all $t \geq t_0$.

3. Main Results

Theorem 1.

Suppose

(3) $$\int_{t_0}^{\infty} \frac{dt}{r(t)} < \infty ,$$

(4) $$\int_{t_0}^{\infty} |a(t)|\rho(t)dt < \infty$$

and

(5) $$\int_{t_0}^{\infty} |b(t)|\rho(t)dt < \infty ,$$

$$\text{where} \qquad \rho(t) = \int_{t}^{\infty} \frac{ds}{r(s)} .$$

Then all bounded oscillatory solutions of (2) approach zero as $t \to \infty$.

Proof. Let $y(t)$ be a bounded oscillatory solution of the equation (2) and

(6) $$\limsup_{t\to\infty} |y(t)| = d_1 > 0 .$$

Then there exists $K > 1$ such that $|y(t)| < K, \qquad |y(g(t))| < K$ for all $t \geq t_0$.
Put $L = max_{z\in[-K,K]}|f(z)|$. Let $T_1 \geq T \geq t_0$ be such that $y(T_1) = 0$, $g(t) \geq T$ for all $t \geq T_1$ and

(7) $$\int_{T}^{\infty} |a(t)|\rho(t)dt < \frac{d}{4L} ,$$

$$\int_T^\infty |b(t)|\rho(t)dt \; < \; \frac{d}{4}\,, \tag{8}$$

where $0 < d < d_1$. It follows from (6) that there exist numbers $x_2 > x_0 > x_1 \ge T_1$ such that $y(x_1) = y(x_2) = 0$, $|y(x_0)| = M > d$, $|y(x)| \le M$ for all $t \in [x_1, x_2]$. Integrating (2) from x_0 to $t \in [x_0, x_2]$ we have

$$r(t)y'(t) + \int_{x_0}^{t} a(x)f(y(g(x)))dx = \int_{x_0}^{t} b(x)dx \tag{9}$$

hence

$$-y(x_0) = -\int_{x_0}^{x_2} \frac{1}{r(t)} \int_{x_0}^{t} a(x)f(y(g(x)))dxdt + \int_{x_0}^{x_2} \frac{1}{r(t)} \int_{x_0}^{t} b(x)dxdt,$$

and

$$M \le \int_{x_0}^{x_2} \frac{1}{r(t)} \int_{x_0}^{t} |a(x)||f(y(g(x)))|dxdt + \int_{x_0}^{x_2} \frac{1}{r(t)} \int_{x_0}^{t} |b(x)|dxdt. \tag{10}$$

As $|y(g(x))| \le K$ for all $x \in [x_0, x_2]$, from relation (10) interchanging integration order and utilizing (7), (8) we have

$$d \le M \le \frac{d}{4} + \frac{d}{4}\,.$$

This contradiction ($d \ne 0$) completes the proof of the theorem.

Theorem 2.

Let the assumptions of the Theorem 1 be satisfied. Furthermore we suppose there exists a positive constant α such that

$$|f(u)| \le \alpha|u|\,. \tag{11}$$

Then all oscillatory solutions of (2) approach zero as $t \to \infty$.

Proof. Since the conditions of the Theorem 1 are satisfied, we only need to prove that the equation (2) has not any unbounded oscillatory solution.

Let $y(t)$ be an oscillatory solution of the equation (2) such that

$$\limsup_{t\to\infty} |y(t)| = \infty\,. \tag{12}$$

Let $T_1 \geq T \geq t_0$ be such that $y(T_1) = 0$, $g(t) \geq T$ for every $t \geq T_1$ and

(13)
$$\int_T^\infty |a(t)|\rho(t)dt < \frac{1}{3\alpha}$$

and

(14)
$$\int_T^\infty |b(t)|\rho(t)dt < \frac{1}{3} .$$

Since $y(t)$ satisfies (12), there are numbers $x_2 \geq x_0 \geq x_1 \geq T_1$ such that $y(x_1) = y(x_2) = 0$, $|y(t)| \leq M$, $|y(g(t))| \leq M$ for $t \in [x_1, x_2]$, $M = |y(x_0)|$, $M > 1$.

Then from (10) interchanging integration order and utilizing the condition (11) we get

$$M \leq \alpha \int_{x_0}^{x_2} |a(t)|\rho(t)|y(g(t))|dt + \int_{x_0}^{x_2} |b(t)|\rho(t)dt ,$$

which gives

$$1 \leq \frac{1}{3} + \frac{1}{3M} .$$

This contradiction completes the proof of the theorem.

Corollary 1.

Suppose (3), (11) and

(15)
$$\int^\infty |a(t)|dt < \infty ,$$

(16)
$$\int^\infty |b(t)|dt < \infty .$$

Then all oscillatory solutions of the equation (2) approach zero as $t \to \infty$.

Proof. The proof follows from the Theorem 2 since with respect to (15) and (16) the conditions (4) and (5) are fulfilled.

Remark 1.

If $f(u) = u$, the Theorem 2 is identical with the Theorem 9, as well as the Corollary 1 with the Theorem 1 in the paper [4]. We remark that the proof of the Theorem 1 as well as that of the Theorem 9 is not quite correct, since generally we cannot affirm the existence of the numbers $T' > T_2 > T_1 > T$ such that $y(T_2) = 0$ and $max\{|y(t)| : T \leq t \leq T'\} = |y(T')| > d > 0$ (see [1] p. 189) for every oscillatory function $y(t)$ such that

$$\limsup_{t\to\infty} |y(t)| > d > 0 \ .$$

Namely, in the concrete, if we have e.g.

$$y(t) = \frac{1}{t} + \sin t \ ,$$

then the condition is not satisfied.

It applies only in the case when $\limsup_{t\to\infty} |y(t)| = \infty$, so in the paper [4] there is essentially only the proof of the assertion, that all oscillatory solutions of the equation (1) are bounded.

As it is seen from the assertion of our Theorem 2 and Corollary 1 the assertions of the Theorem 1 and 9 in the paper [4] are correct, and the gap in proofs can be corrected.

Theorem 3.

Suppose $a(t) = a_1(t) + a_2(t)$, $a_1(t) > 0$, $\frac{a_2(t)}{a_1(t)}$ is bounded for $t \to \infty$,

$$\int^{\infty} \frac{1}{r(t)} dt = \infty \ , \tag{17}$$

$$\lim_{t\to\infty} \frac{|b(t)|}{a_1(t)} = \infty \tag{18}$$

and

$$\int^{\infty} a_1(t) dt = \infty \ . \tag{19}$$

Then all solutions of the equation (2) are unbounded.

Proof. From the equation (2) we have

$$\frac{(r(t)y'(t))'}{a_1(t)} + \left(1 + \frac{a_2(t)}{a_1(t)}\right) f(y(g(t))) = \frac{b(t)}{a_1(t)} \ ,$$

which gives

$$\frac{(r(t)y'(t))'}{a_1(t)} \geq \frac{|b(t)|}{a_1(t)} - (1+k_1)|f(y(g(t)))| \ . \tag{20}$$

If $y(t)$ is a bounded solution, then from the last relation we get that there exists a constant $K_1 > 0$ such that
$(r(t)y'(t))' \geq K_1 a_1(t)$ for every $t \geq t_1$.
From this relation we get

$$r(t)y'(t) \geq r(t_1)y'(t_1) + K_1 \int_{t_1}^{t} a_1(s)ds \ , \tag{21}$$

which gives a contradiction with respect to (19) and (17). The proof is complete.

Remark 2.

If $r(t)$ is a bounded function, $a_1(t) > \rho > 0$ and $f(u) = u$ then we get the Theorem 6 in the paper [1].

Theorem 4.

Let the assumptions of the Theorem 3 be satisfied, whereby instead of (17) and (19) there is (3) and

$$\int^{\infty} a_1(t)\rho(t)dt = \infty \ . \tag{22}$$

Then all solutions of (2) are unbounded.

Proof. If $y(t)$ is a bounded solution of the equation (2), then from the relation (21) we get

$$y(t) \geq y(t_1) + r(t_1)y'(t_1) \int_{t_1}^{t} \frac{1}{r(s)} ds + K_1 \int_{t_1}^{t} a_1(s)\rho(s)ds$$

which gives a contradictions with respect to (3) and (22).

Corollary 2.

Let the assumptions of the Theorem 3 be satisfied, whereby instead of (18) we suppose

$$\liminf_{t\to\infty} \frac{|b(t)|}{a_1(t)} = d > 0 \ . \tag{23}$$

Then the equation (2) has not any solution approaching zero as $t \to \infty$.

Proof. Let $y(t)$ be a solution of the equation (2) such that $\lim_{t\to\infty} y(t) = 0$. Then from the relation (20), with respect to properties of function $f(u)$ we get

$$(r(t)y'(t))' \geq \frac{d}{2}a_1(t) \qquad \text{for every} \qquad t \geq t_1 \; .$$

We get a contradiction analogous to the Theorem 3. It is clear that the Corollary 3 applies.

Corollary 3.

Let the assumptions of the Theorem 4 be satisfied, whereby we suppose (23) instead (18). Then the equation (2) has not any solution approaching zero as $t \to \infty$.

Theorem 5.

Suppose (4) and

$$\int^{\infty} b(t)\rho(t)dt = \infty \; . \tag{24}$$

Then all solutions of the equation (2) are unbounded.

Proof. Let $y(t)$ be a bounded solution of the equation (2). Then from the equation (2) we have

$$r(t)y'(t) + |r(t_1)y'(t_1)| + K_2 \int_{t_1}^{t} |a(s)|ds \geq \int_{t_1}^{t} b(s)ds \; , \tag{25}$$

hence

$$y(t) - y(t_1) + |r(t_1)y'(t_1)| \int_{t_1}^{t} \frac{1}{r(s)}ds + K_2 \int_{t_1}^{t} \frac{1}{r(s)} \int_{t_1}^{s} |a(x)|dxds \geq \int_{t_1}^{t} \frac{1}{r(s)} \int_{t_1}^{s} b(x)dxds \; .$$

From the last relation we have

$$\lim_{t\to\infty} y(t) \;\geq\; K_3 + \int_{t_1}^{\infty} b(t)\rho(t)dt \; ,$$

which is a contradiction since $y(t)$ is bounded.

Corollary 4.

Suppose (15), (17) and

$$\int^{\infty} b(t)dt = \infty \,. \tag{26}$$

Then all solutions of the equation (2) are unbounded.

Proof. If $y(t)$ is a bounded solution of the equation (2), then from the relation (25), with respect to (26) and (15) we get, that $r(t)y'(t) \geq K_4 > 0$ for every $t \geq t_1$, which gives a contradiction with respect to (17), since $y(t)$ is bounded. The proof is complete.

Theorem 6.

Suppose (3), (4), (11) and $b(t) \geq 0$. Then (5) is necessary and sufficient condition in order that all oscillatory solutions of the equation (2) may approach zero as $t \to \infty$.

Proof. If (5) applies, then the conditions of the Theorem 2 are satisfied, and so all oscillatory solutions of the equation (2) approach zero as $t \to \infty$.

Now we are going to prove that (5) is a necessary condition. So all oscillatory solutions of the equation (2) approach zero as $t \to \infty$ and (5) does not apply i.e. let

$$\int^{\infty} b(t)\rho(t)dt = \infty \,.$$

Then the conditions of the Theorem 5 are satisfied and so all solutions of the equation (2) are unbounded which gives a contradiction. This proves the Theorem 6.

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Vincent Šoltés
Mathematics Department
Technical University Košice
S L O V A K I A