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ON THE REMAINDER IN AN APPROXIMATION FORMULA BY FAVARD-SZASZ TYPE OPERATOR FOR FUNCTIONS OF TWO VARIABLES

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INTRODUCTION. A.Jakimovski and D.Leviatan [3] have introduced a Favard-Szasz type operator, by means of Appell polynomials. One considers $g(z) = \sum_{n=0}^{\infty} a_n z^n$ an analytic function in the disk $|z| < R, R > 1$, where $g(1) \neq 0$. Define the Appell polynomials $p_k(x), k \geq 0$ by

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k \tag{1}$$

One defines a sequence of linear operators, associated to each function f defined in $[0, \infty)$, as follows:

$$(P_n f)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \tag{2}$$

The case $g(z) = 1$ yields the classical operators of Favard-Szasz [2], [4].

$$(S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)$$

II. In [1] we have extended to two variables this operator, as follows.

Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $h(z) = \sum_{n=0}^{\infty} b_n z^n$ be two analytic functions in the disk $|z| < R, R > 1$ and let us denoting by p_k *and* $q_k, k = 0, 1, \ldots$ the Appell polynomials generated by g and h, by the relations :

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k \;\; and \;\; h(v)e^{vx} = \sum_{k=0}^{\infty} q_k(x)v^k \tag{3}$$

To each function f defined in $D = [0, \infty) \times [0, \infty)$ is associated the operator $P_{m,n}$ according to the formula:

$$(P_{m,n} f)(x, y) = \frac{e^{-mx}e^{-ny}}{g(1)h(1)} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} p_k(mx) q_i(ny) f\left(\frac{k}{m}, \frac{i}{n}\right) \tag{4}$$

where m and n are natural numbers.

If $g(z) = 1$ *and* $h(z) = 1$, from (4) we obtain the well known Favard-Szasz operator for two variables:

$$(S_{m,n} f)(x, y) = e^{-mx}e^{-ny} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{(mx)^k}{k!} \frac{(ny)^i}{i!} f\left(\frac{k}{m}, \frac{i}{n}\right)$$

We suppose that p_k *and* q_k , k=0,1,... are positive and so, the operator $P_{m,n}$ is positive.

Let E_2 be the space of functions $f:[0, \infty) \times [0, \infty) \to f:[0, \infty) \times [0, \infty) \to R$ of exponential type. More precise, $f \in E_2$ if and only if there are two finite constants A and B with the property : $|f(x, y)| \leq e^{Ax + By}$

In [1] we have proved

Theorem A:

If $f \in C(D) \cap E_2$, *then* $\lim_{m,n\to\infty} (P_{m,n}f)(x, y) = f(x, y)$, *the convergence being uniform in each compact* $[0, a] \times [0, b]$.

The values of the operator for the test functions $e_{0,0}(t, \tau) = 1, e_{1,0}(t, \tau) = t, e_{0,1}(t, \tau) = \tau$ *and* $e_{2,2}(t, \tau) = t^2 + \tau^2$ are

$$\begin{aligned}
&(P_{m,n}e_{0,0})(x, y) = 1 \\
&(P_{m,n}e_{1,0})(x, y) = x + \frac{1}{m}\frac{g'(1)}{g(1)} \\
&(P_{m,n}e_{0,1})(x, y) = y + \frac{1}{n}\frac{h'(1)}{h(1)} \\
&(P_{m,n}e_{2,2})(x, y) = x^2 + y^2 + \frac{x}{m}\left(1 + 2\frac{g'(1)}{g(1)}\right) + \frac{y}{n}\left(1 + 2\frac{h'(1)}{h(1)}\right) + \\
&\qquad + \frac{1}{m^2}\frac{g''(1) + g'(1)}{g(1)} + \frac{1}{n^2}\frac{h''(1) + h'(1)}{h(1)}
\end{aligned} \tag{5}$$

We next turn to the task of stabilising an asymptotic estimate of the remainder in the approximation formula

$$(R_{m,n}f)(x, y) = f(x, y) - (P_{m,n}f)(x, y)$$

which corresponds to Voronovskaia theorem about Bernstein polynomials. Let us denoting $D_2 = [0, a] \times [0, b]$.

Theorem

If the function f is defined and bounded in D_2 *and at an interior point (x,y) of* D_2 *the second total differential* $d^2 f(x, y)$ *exists, then we have the asymptotic formula*

$$\begin{aligned}
-(R_{m,n}f)(x, y) &= f'_x(x, y)\frac{1}{m}\frac{g'(1)}{g(1)} + f'_y(x, y)\frac{1}{n}\frac{h'(1)}{h(1)} + \frac{1}{2}f''_{xx}(x, y)\left(\frac{x}{m} + \frac{1}{m^2}\frac{g''(1) + g'(1)}{g(1)}\right) + \\
&+ \frac{1}{2}f''_{yy}\left(\frac{y}{n} + \frac{1}{n^2}\frac{h''(1) + h'(1)}{h(1)}\right) + f''_{xy}\frac{1}{mn}\frac{g'(1)}{g(1)}\frac{h'(1)}{h(1)} + \rho_{m,n}(x, y)
\end{aligned}$$

where $\lim_{m,n\to\infty} \rho_{m,n}(x, y) = 0$

Proof: Let $(t, \tau) \in D_2 = [0, a] \times 0, b]$. One knows that there is a function $G(t, \tau)$, *defined on* D_2 such that when $t \to x$ *and* $\tau \to y$ *we have* $G(t, \tau) \to 0$ *and* $f(t, \tau)$ may be expanded by Taylor's formula :

$$\begin{aligned}
f(t, \tau) &= f(x, y) + (t - x)f'_x(x, y) + (\tau - y)f'_y(x, y) + \frac{1}{2}(t - x)^2 f^2_{xx}(x, y) + \\
&+(t - x)(\tau - y)f''_{xy}(x, y) + \frac{1}{2}(\tau - y)^2 f''_{yy}(x, y) + \left[(t - x)^2 + (\tau - y)^2\right]G(t, \tau)
\end{aligned}$$

If we replace here $t = \frac{k}{m}$ *and* $\tau = \frac{i}{n}$, multiply by $\frac{e^{-mx}}{g(1)}\frac{e^{-ny}}{h(1)} p_k(mx)q_i(ny)$, sum over k and i, we obtain :

$$\begin{aligned}
&-(R_{m,n}f)(x, y) = (P_{m,n}f)(x, y) - f(x, y) = \\
&= \frac{e^{-mx}}{g(1)}\frac{e^{-ny}}{h(1)}\sum_{i,k=0}^{\infty} p_k(mx)q_i(ny)\left\{\left(\frac{k}{m} - x\right)f'_x(x, y) + \left(\frac{i}{n} - y\right)f'_y(x, y) + \right. \\
&+ \frac{1}{2}\left(\frac{k}{m} - x\right)^2 f''_{xx}(x, y) + \left(\frac{k}{m} - x\right)\left(\frac{i}{n} - y\right)f''_{xy}(x, y) + \frac{1}{2}\left(\frac{i}{n} - y\right)^2 f''_{yy}(x, y) +
\end{aligned}$$

$$+\left[\left(\frac{k}{m}-x\right)^2+\left(\frac{i}{n}-y\right)^2\right]G\left(\frac{k}{m},\frac{i}{n}\right)\}$$

By making use of the values of the operator $P_{m,n}$ for the test functions , we find :

$$-(R_{m,n}f)(x,y) = f_x^{'}(x,y)\frac{1}{m}\frac{g'(1)}{g(1)} + f_y^{'}(x,y)\frac{1}{n}\frac{h'(1)}{h(1)} +$$

$$\frac{1}{2}f_{xx}^{''}(x,y)\left(\frac{x}{m}+\frac{1}{m^2}\frac{g''(1)+g'(1)}{g(1)}\right)+\frac{1}{2}f_{yy}^{''}(x,y)\left(\frac{y}{n}+\frac{1}{n^2}\frac{h''(1)+h'(1)}{h(1)}\right)+$$

$$+f_{xy}^{''}(x,y)\frac{1}{mn}\frac{g'(1)}{g(1)}\frac{h'(1)}{h(1)}+\rho_{m,n}(x,y)$$

where we have denoted

$$\rho_{m,n}(x,y) = \frac{e^{-mx}}{g(1)}\frac{e^{-ny}}{h(1)}\sum_{i,k=0}^{\infty}p_k(mx)q_i(ny)\left[\left(\frac{k}{m}-x\right)^2+\left(\frac{i}{n}-y\right)^2\right]G\left(\frac{k}{m},\frac{i}{n}\right)$$

Next, we will prove that $\lim_{m,n\to\infty}\rho_{m,n}(x,y)=0$. Since $G(t,\tau)\to 0$ *when* $t\to x$ *and* $\tau\to y$, it follows that for every positive ε , there correspond the positive numbers δ_1 *and* δ_2 δ_1 *and* δ_2 such that $|G(t,\tau)|<\varepsilon$ whenever $|t-x|\le\delta_1$ *and* $|\tau-y|\le\delta_2$. If we set $t=\frac{k}{m}$ *and* $\tau=\frac{i}{n}$, we have

$$\left|G\left(\frac{k}{m},\frac{i}{n}\right)\right|<\varepsilon \text{ whenever } \left|\frac{k}{m}-x\right|\le\delta_1 \text{ and } \left|\frac{i}{n}-y\right|\le\delta_2$$

On the other hand, there is a constant M>0 such that $\left[\left(\frac{k}{m}-x\right)^2+\left(\frac{i}{n}-y\right)^2\right]\left|G\left(\frac{k}{m},\frac{i}{n}\right)\right|<M$

Let us denote $\omega_{k,i}(x,y)=p_k(mx)q_i(ny)\left[\left(\frac{k}{m}-x\right)^2+\left(\frac{i}{n}-y\right)^2\right]\left|G\left(\frac{k}{m},\frac{i}{n}\right)\right|$

If we divide the set of indices (k,i) into four classes :

$$I_1=\left\{(k,i):\left|\frac{k}{m}-x\right|\le\delta_1,\left|\frac{i}{n}-y\right|\le\delta_2\right\}$$

$$I_2=\left\{(k,i):\left|\frac{k}{m}-x\right|\le\delta_1,\left|\frac{i}{n}-y\right|>\delta_2\right\}$$

$$I_3=\left\{(k,i):\left|\frac{k}{m}-x\right|>\delta_1,\left|\frac{i}{n}-y\right|\le\delta_2\right\}$$

$$I_4=\left\{(k,i):\left|\frac{k}{m}-x\right|>\delta_1,\left|\frac{i}{n}-y\right|>\delta_2\right\}$$

we can write

$$|\rho_{m,n}(x,y)|\le\frac{e^{-mx}}{g(1)}\frac{e^{-ny}}{h(1)}\sum_{\nu=1}^{4}\sum_{(k,i)\in I_\nu}\omega_{k,i}(x,y)$$

Next, we may proceed in the same way as in the case of one variable. For instance :

$$\frac{e^{-mx}}{g(1)}\frac{e^{-ny}}{h(1)}\sum_{(k,i)\in I_2}\omega_{k,i}(x,y)=$$

$$=\frac{e^{-mx}}{g(1)}\frac{e^{-ny}}{h(1)}\sum_{\left|\frac{k}{m}-x\right|\le\delta_1}\sum_{\left|\frac{i}{n}-y\right|>\delta_2}p_k(mx)q_i(ny)\left[\left(\frac{k}{m}-x\right)^2+\left(\frac{i}{n}-y\right)^2\right]\left|G\left(\frac{k}{m},\frac{i}{n}\right)\right|$$

$$\le\frac{e^{-mx}}{g(1)}\frac{e^{-ny}}{h(1)}M\sum_{\left|\frac{i}{n}-y\right|>\delta_2}q_i(ny)\sum_{\left|\frac{k}{m}-x\right|\le\delta_1}p_k(mx)\le\frac{e^{-mx}}{g(1)}\frac{e^{-ny}}{h(1)}M\sum_{\left|\frac{i}{n}-y\right|>\delta_2}q_i(ny)\sum_{k=0}^{\infty}p_k(mx)=$$

$$= M \frac{e^{-ny}}{h(1)} \sum_{\left|\frac{i}{n}-y\right|>\delta_2} q_i(ny)$$

If $\delta_2 < \left|\frac{i}{n} - y\right|$, it results that $1 \le \frac{1}{\delta_2^2}\left(\frac{i}{n} - y\right)^2$ and we arrive to the inequality :

$$\frac{e^{-mx}}{g(1)} \frac{e^{-ny}}{h(1)} \sum_{k\in I_2} \omega_{k,i}(x,y) \le M \frac{e^{-ny}}{h(1)} \sum_{i=0}^{\infty} \frac{1}{\delta_2^2}\left(\frac{i}{n} - y\right)^2 q_i(ny) = M \frac{1}{\delta_2^2}\left(\frac{y}{n} + \frac{1}{n^2} \frac{h''(1) + h'(1)}{h(1)}\right)$$

Thus, we get to :

$$|\rho_{m,n}(x,y)| \le \varepsilon\left(\frac{x}{m} + \frac{1}{m^2}\frac{g''(1)+g'(1)}{g(1)} + \frac{y}{n} + \frac{1}{n^2}\frac{h''(1)+h'(1)}{h(1)}\right) +$$

$$+ M\left[\frac{1}{\delta_1^2}\left(\frac{x}{m} + \frac{1}{m^2}\frac{g''(1)+g'(1)}{g(1)}\right) + \frac{1}{\delta_2^2}\left(\frac{y}{n} + \frac{1}{n^2}\frac{h''(1)+h'(1)}{h(1)}\right) +\right.$$

$$\left. + \frac{1}{\delta_1^2\delta_2^2}\left(\frac{x}{m} + \frac{1}{m^2}\frac{g''(1)+g'(1)}{g(1)}\right)\left(\frac{y}{n} + \frac{1}{n^2}\frac{h''(1)+h'(1)}{h(1)}\right)\right]$$

Finally, we can write :

$$|\rho_{m,n}(x,y)| \le \varepsilon\left(\frac{x}{m} + \frac{y}{n}\right) + \frac{1}{m^2}\frac{g''(1)+g'(1)}{g(1)}\left(\varepsilon + \frac{M}{\delta_1^2}\right) + \frac{1}{n^2}\frac{h''(1)+h'(1)}{h(1)}\left(\varepsilon + \frac{M}{\delta_2^2}\right) +$$

$$+ \frac{M}{\delta_1^2\delta_2^2}\left(\frac{x}{m} + \frac{1}{m^2}\frac{g''(1)+g'(1)}{g(1)}\right)\left(\frac{y}{n} + \frac{1}{n^2}\frac{h''(1)+h'(1)}{h(1)}\right).$$

Thus, we reach the conclusion that $\lim_{m,n\to\infty} \rho_{m,n}(x,y) = 0$. This completes the proof.

Remark: In the case $g(z) = 1$, $h(z) = 1$ *and* $m = n$ we obtain for the Favard-Szasz operators:

$$-(R_m f)(x,y) = \frac{x}{2m} f''_{xx}(x,y) + \frac{y}{2m} f''_{yy}(x,y) + \theta_m(x,y)$$

where, for $\theta_m(x,y) = e^{-mx}e^{-my} \sum_{i,k=0}^{\infty} \frac{(mx)^k}{k!}\frac{(my)^i}{i!}\left[\left(\frac{k}{m} - x\right)^2 + \left(\frac{i}{m} - y\right)^2\right] G\left(\frac{k}{m}, \frac{i}{m}\right)$

we have $|\theta_m(x,y)| \le \varepsilon\left(\frac{x}{m} + \frac{y}{m}\right) + \frac{1}{\delta_1^2\delta_2^2}\frac{xy}{m^2}$ and $\lim_{m\to\infty} \theta_m(x,y) = 0$.

F.Stancu [5] and A.Lupas [4] studied the remainder in the approximation formula of a function of two variables by the operators Favard-Szasz.

REFERENCES

1. Alexandra Ciupa, Operatori generalizati de tip Favard-Szasz pentru aproximarea funcţiilor de două variabile, Buletin St.Univ. Baia-Mare, Seria B, Matematică - Informatică , vol. IX (1993) , 19-24.
2. J.Favard, Sur les multiplicateurs d'interpolation , J.Math. Pures Appl. 23 (1944) , 219-247.
3. A.Jakimovski, D.Leviatan, Generalized Szasz operators for the approximation in infinite interval, Mathematica, (Cluj), 34, (1969), 97-103.
4. A.Lupaş, Some properties of the linear pozitive operators (I), Mathematica, 9 (32), 1, (1967), 77-83.
5. F.Stancu, Asupra restului în formula de aproximare prin operatorii lui Mirakyan de una şi două variabile, Analele Şt.Univ. "Al.I.Cuza" , Iaşi, 14(1968) , 415-422.
6. O.Szasz, Generalization of S.Bernstein's polynomials to the infinite interval, J.Res. Nat. Bur. Standards, 45 (1950), 239-245.

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