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ON A HOMEOMORPHISM THEOREM

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1. INTRODUCTION

In most applications of the fixed point theorems the ambient complete metric space is actually a Banach space. Because of this richer structure, a generalized contraction principle leads to a result especially useful in applications, as in the case of contraction mapping principle [6].

In this paper we give a generalization of the following homeomorphism theorem due to DUGUNDJI and GRANAS [6], Corollary 2.2.

THEOREM 1. *Let X be a Banach space and $f: X \to X$ a contraction, i.e. a mapping for which there exists a constant a, $0 \le a < 1$, such that*

$$\|f(x)-f(y)\| \le a \cdot \|x-y\| \text{ , for each } x,y \in X \text{ .}$$

Then $1_X - f$ is a homeomorphism of X onto itself.

To this end we need some definitions, examples and results from BERINDE,V. [1]-[5], RUS,A.I. [7], [9].

2. GENERALIZED CONTRACTIONS

Definition 1. (BERINDE [1]) A function $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ is called *(c)-comparison function* if the following two conditions are satisfied:

(c_1) φ is monotone increasing;

(c_2) There exist two numbers, k_0 and α, $0 \le \alpha < 1$, and a convergent series with nonnegative terms $\sum_{k=1}^{\infty} a_k$, such as

$$\varphi^{k+1}(t) \le \alpha\varphi^{k}(t) + a_k \text{ , for each } t\in\mathbb{R}_+ \text{ and } k\ge k_0 \text{ ,}$$

(φ^k stands for the k iterate of φ).

LEMMA 1. (BERINDE [1], [5]) *If φ is a (c)-comparison function then*

(c_3) $\varphi(t)<t$, for each $t>0$;

(c_4) φ is continuous at 0;

(c_5) The series

$$\sum_{k=0}^{\infty} \varphi^{k}(t) \tag{1}$$

converges for each $t\in\mathbb{R}_+$;

(c_6) The sum of the series (1), denoted $s(t)$, is monotone increasing and continuous at 0;

(c_7) $(\varphi^n(t))_{n\in N}$ converges to 0, as $n\to\infty$, for each $t\in\mathbb{R}_+$.

Remark. A function φ: $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfies (c_1) and (c_7) is called *comparison function* (see RUS,A.I.[8]).

Exemple 1. If a∈0,1), then φ: $\mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t)=at$, $t\in\mathbb{R}_+$, is a (c)-comparison function, hence φ is a comparison function too. But there exist comparison functions which are not (c)-comparison functions. Indeed, $\varphi:\mathbb{R}_+\to\mathbb{R}_+$, $\varphi(t)=\frac{t}{1+t}$, $t\in\mathbb{R}_+$, is a comparison function which is not a (c)-comparison function.

The importance of (c)-comparison functions consists in the fact that in a genralized fixed point principle with respect to a (c)-comparison function, the error estimates of the fixed point may be given in a similar way to the one in the classical contraction mapping principle, see BERINDE [1]-[5], as shown by

THEOREM 2. *Let (X,d) be a complete metric space and f: X → X a φ-contraction, that is, a mapping for which there exists a comparison function φ such that*

$$d(f(x),f(y)) \le \varphi(d(x,y)) \text{ , for each } x,y\in X \text{ .}$$

If φ is a (c)-comparison function, then

1) $F_f=\{x^*\}$ $(F_f=\{x\in X/f(x)=x\})$;

2) If (x_n) *is the sequence of succesive approximations, defined by* $x_n=f(x_{n-1}), n\geq 1$ *and* $x_0\in X$ *, arbitrary taken, then* (x_n) *converges to* x^* *.*

3. The error estimates is given by

$$d(x_n,x^*)\leq s(d(x_n,x_{n+1})), n\geq 1 .$$

Remark. For $\varphi=\alpha t$, $0\leq\alpha<1$, from theorem 2 we obtain the classical contraction mapping principle. If φ is a comparison function rather than a (c)-comparison function, then only the statements 1) and 2) of theorem 3 hold.

Definition 2. (RUS,A.I. [8]) Let (X,d) be a metric space. An operator f: X → X satisfying the following conditions

(i) There exists $x^*\in X$, such that $F_f=\{x^*\}$;

(ii) The sequence $(f^n(x_0))_{n\in N}$ converges to x^* , for each $x_0\in X$, is called *Picard operator*.

If F: X → X is a Picard operator and the corresponding field $f=1_X-F$ is a homeomorphism, then F is called *Banach operator*.

Remark. Theorem 2 shows that any φ-contraction is a Picard operator.

3. HOMEOMORPHISM THEOREMS.

The generalized contraction principle (Theorem 2) has an useful local version that involves an open ball B in a complete metric space X and a φ-contraction of B into X which does not displace the center of the ball to far:

LEMMA 2. *Let (X,d) be a complete metric space and* $B(a,r)=\{x\in X/d(x,a)<r\}$ *.*

Let $f:B\to X$ *be a* φ*-contraction, with* φ *a given comparison function. If the following condition*

$$d(f(a),a) < r-\varphi(r)$$

is satisfied, then f has a fixed point.

Proof. Since φ is a comparison function, φ is a continuous at 0 and hence the function $h:\mathbb{R}_+\to\mathbb{R}_+$, $h(t)=t-\varphi(t)$ is continuous at 0 too and h(0)=0. This means that we can choose ε<r such that

$$d(f(a),a)\le\varepsilon-\varphi(\varepsilon)<r-\varphi(r).$$

Let's denote $\overline{D}(a,\varepsilon)=\{x\in X/d(a,x)\le\varepsilon\}$.
The conclusion of lemma follows now from Theorem 2 applied for the mapping g, the restriction of f to $\overline{D}(a,\varepsilon)$.

Remark. For $\varphi(t)=\alpha t$, $0\le\alpha<1$, from Lemma 2 we obtain Corollary 1.2, DUGUNDJI and GRANAS [6].

Definition 3. Let E be a subset of a Banach space X. Given a map F:E → X, the map x ↦ x-F(x) of E into X is called the *field associated with f*, and is denoted by the corresponding lower-case letter: f(x)=x-F(x). The field f:E → X determinated by a generalized contractive map F:E → X is called a *generalized contractive field*.

THEOREM 3. (Invariance of domain for generalized contractive fields). *Let X be a Banach space, E⊂X open, and F: E → X a φ-contraction. Let f:E → X be the associated field, f(x)=x-F(x). Then*

(a) f:E→ X is an open mapping;in particular,f(E) is open in E;

(b) f:E → f(E) is a homeomorphism.

Proof. To show that f is an open mapping, it is enough to establish that for any a∈E, if B(a,r)⊂E, then

$$B[f(a),r-\varphi(r)]\subset f[B(a,r)].$$

For this purpose, choose any $y_0\in B(f(a),r-\varphi(r))$ and define G:B(a,r)→ →X by

$$G(y)=y_0+F(y)\ ,\ y\in B(a,r).$$

Then G is a φ-contraction and

$$\|G(u)-u\|=\|y_0+F(u)-u\|=\|y_0-f(u)\|\le r-\varphi(r).$$

So, by Lemma 3, there is an $u_0 \in B(u,r)$ with $u_0=y_0+F(u_0)$, that is, $f(u_0)=y_0$. Thus,

$$b[f(u),r-\varphi(r)] \subset f[B(u,r)],$$

so $f:E \to X$ is an open mapping and, in particular, $f(E)$ is open in X. To prove (b), we observe that if $u,v \in U$, then

$$\|f(u)-f(v)\| = \|u-v-(F(u)-F(v))\| \geq \|u-v\| - \|F(u)-F(v)\| .$$

But

$$\|F(u)-F(v)\| \leq \varphi(\|u-v\|),$$

hence

$$\|f(u)-f(v)\| \geq u-v\| - \varphi(\|u-v\|),$$

so that f is injective. Indeed, if $f(u)=f(v)$, then we obtain

$$\varphi(\|u-v\|) = \|u-v\|,$$

which implies, in virtue of (c_3) and (c_4),

$$\|u-v\| = 0,$$

that is, $u=v$.

By the same condition (c_3) it results that any φ-contraction is a continuous mapping, hence F is continuous.
This means $f: U \to f(U)$ is a continuous open bijection and therefore it is a homeomorphism.
The proof is now complete.

THEOREM 4. *Let X be a Banach space and $F:X \to X$ a φ-contraction. Then the associated field $f=I-F$ is a homeomorphism of X onto itself.*

Proof. In view o theorem 3 we need to show only that $f(X)=X$, that is, f is surjective. Given $y_0 \in X$, define $G:X \to X$, by

$$x \to y_0+F(x) .$$

Then G is a φ-contraction, so it has a fixed point $x_0=y_0+F(x_0)$, that is, $y_0=f(x_0)$.

Remark. Theorem 4 shows that any φ-contraction defined on a Banach space is a Banach operator. This result consists an answer to the problem 9.2.1(a), RUS,A.I. [8]. Finally, let's observe that all results of this paper hold if we replace the metric space by a K-metric space and the Banach space by a K-normed Banach space, see BERINDE,V. [3]-[5].

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