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UNIFORM BOUNDEDNESS AND STABILITY OF SOLUTIONS OF A NONLINEAR TWO DIMENSIONAL DIFFERENTIAL SYSTEM

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Abstract

There are introduced some sufficient conditions for a solution $(x(t), y(t))$ of the system

$$x' = y$$
$$y' = -f_1(t,x,y)f_2(x)y - g_1(t,x)g_2(y) - h(t,x,y) - e(t,x,y)$$

to be uniformly bounded. Also there are found some sufficient conditions for the convergence of all solutions $(x(t), y(t))$ of the system (1) to be origin as $t \to \infty$.

We consider the two dimensional differential system of the form

(1)
$$x' = y$$
$$y' = -f_1(t,x,y)f_2(x)y - g_1(t,x)g_2(y) - h(t,x,y) - e(t,x,y)$$

where $f_1, h, e \in C(D_2)$, $g_1, \frac{\partial g_1}{\partial t} \in C(D_1)$, $f_2, g_2 \in C(R_1)$, $R_1 = (-\infty, \infty)$, $R_2 = R_1 \times R_1$, $I = [0, \infty)$, $D_1 = I \times R_1$ and $D_2 = I \times R_2$.

Let us define the following functions:

$$G_1(t,x) = \int_0^x g_1(t,s)ds \quad \text{in} \quad D_1,$$

$$F_2(x) = \int_0^x sf_2(s)ds \quad \text{and} \quad G_2(y) = \int_0^y \frac{s}{g_2(s)}ds \quad \text{in} \quad R_1.$$

In what follows, the following conditions will be required:

(2) $$xg_1(t,x) > 0 \quad \text{for } x \neq 0 \text{ and } t \in I;$$

(3) There exists a positive function $f_3 \in C(R_2)$, such that
$f_1(t,x,y) \geq f_3(x,y)$ in D_2 and
$xyg_2(y) \leq y^2 f_3(x,y)$ for $x, y \in R_1$;

(4) $$G_2(y) \to \infty \text{ for } |y| \to \infty;$$

(5) $$yh(t,x,y) \geq 0 \text{ in } D_2;$$

(6) $$g_2(y) > 0, \quad f_2(x) > 0 \text{ in } R_1.$$

We will use the following definitions and propositions.
Let $\varphi(t) = \varphi(t; t_0, x_0)$ denote a solution of the system

(7) $$x' = f(t,x),$$
$x \in R_n$, $t \in I$, $f(t,x) \in C(I \times R_n)$ through x_0 at $t = t_0$.

Definition 1. *The solutions of* (7) *are uniformly bounded if for any* $(t_0, \alpha) \in I \times R_1$ *there exists* $\beta = \beta(\alpha) > 0$ *such that* $|x_0| \leq \alpha$ *implies* $|\varphi(t; t_0, x_0)| \leq \beta(\alpha)$ *for every* $t \geq t_0$.

Proposition 1. (see [2]) *Let there exist continuous function* $V(t,x)$ *and* $W_i(x)$, $i = 1,2$ *in* $I \times R_n$ *such that the following conditions hold:*

1. $0 < W_1(x) \leq V(t,x) \leq W_2(x)$, $W_1(x) \to \infty$, $|x| \to \infty$;
2. $V'(t,x) = \dfrac{\partial V}{\partial t} + \dfrac{\partial V}{\partial x} f(t,x) \leq 0.$

Then the solutions of (7) *are uniformly bounded for* $t \geq 0$.

Definition 2. (see [2]) *A point* $x \in R_n$ *is called an* ω*-limit point of a solution* $\varphi(t)$ *of the system* (7) *if there exists a sequence* $\{t_n\}_{n=1}^{\infty} \subset I$ *such that* $\varphi(t_n) \to x$, $t_n \to \infty$, $n \to \infty$.

Definition 3. (see [2]) *The set of all* ω*-limit points of solution* $\varphi(t)$ *is called the limit set of a solution* $\varphi(t)$. *This set will be denoted by* $\Omega(\varphi)$.

Definition 4. (see [3]) *A function* $V(t,x)$ *is called uniformly small if there exists a continuous, positively definite function* $W(x)$ *such that* $V(t,x) \leq W(x)$ *in* $I \times R_n$.

Proposition 2. (see [3]) *If there exists a positively definite uniformly small function* $V(t,x)$, *which has a negatively definite derivative with respect to* t, *then the trivial solution of the system* (7) *is uniformly asymptotically stable.*

PROPOSITION 3. (see [2]) *Let the assumptions of Proposition 1 be fulfilled. Moreover, suppose that the function on the right sides of the system (7) are bounded in $I \times Q$, where $Q \subset R_n$ is any compact set and there exists a continuous function $W_3(x)$ such that*

$$V'(t,x) \leq -W_3(x) \leq 0, \quad \text{in } I \times R_n.$$

Then

$$\varphi(t) \to \{x : |x| \leq \beta, W_3(x) = 0\}, \ \ t \to \infty,$$

where the constant β is as in Definition 1..

DEFINITION 5. (see [1]) *The trivial solution of the system* (7) *is called globally asymptotically stable if is asymptotically stable and all solution $\varphi(t;t_0,x_0)$ of the system* (7) *converges to zero as $t \to \infty$.*

THEOREM 1. *Suppose that the assumptions* (2)-(6) *and the following conditions are satisfied:*

$$yc(t,x,y) \geq 0 \quad \text{in } D_2; \tag{8}$$

$$\frac{\partial G_1(t,x)}{\partial t} \leq 0 \quad \text{in } D_1; \tag{9}$$

(10) *There exist functions $p_i \in C(R_1)$, $i = 1,2$ such that*

$$xp_i(x) > 0 \quad (i = 1,2) \quad \text{for } x \neq 0$$

and

$$|p_1(x)| \leq |g_1(t,x)| \leq |p_2(x)| \quad \text{in } D_1,$$

where

$$P_1(x) = \int_0^x p_1(s)ds \to \infty \quad \text{for } |x| \to \infty.$$

Then the solutions of (1) *are uniformly bounded for $t \geq 0$.*

PROOF. For any solution $(x(t), y(t))$ of (1) and a positive constant K we define

$$V(t,x,y) = F_2(x) + G_1(t,x) + G_2(y) + K \quad \text{in } D_2.$$

By (2), (6) and (10) we get

$$0 < W_1(x,y) = P_1(x) + G_2(y) + K \leq V(t,x,y) \leq F_2(x) + P_2(x) + G_2(y) + K$$

(where $P_2(x) = \int_0^x p_2(s)ds$) in D_2. From (4) and (10) it follows

$$W_1(x,y) \to \infty \quad \text{for } |x| \to \infty \text{ and } |y| \to \infty.$$

Differentiate the function $V(t) = V(t, x(t), y(t))$ with respect to t. Using (2), (3), (5), (6) and (9) we obtain

$$(11) \qquad V'(t) \leq \left(xy - f_1(t,x,y)\frac{y^2}{g_2(y)} \right) f_2(x) \leq 0 \quad \text{in } D_2.$$

This means that all conditions of Proposition (1) are fulfilled, therefore the solutions of (1) are uniformly bounded for $t \geq 0$. This completes the proof.

COROLLARY 1. *Let the hypotheses of Theorem 1 hold. Then the trivial solutions of* (1) *is uniformly asymptotically stable.*

PROOF. From (2), (5) and (8) it follows $g_1(t,0) = 0$, $h(t,x,0) = 0$ and $e(t,x,0) = 0$ for $t \in I$ and $x \in R_1$. This means that (1) has the trivial solution. For any solution $(x(t), y(t))$ of (1) we define

$$V(t,x,y) = F_2(x) + G_2(y) + G_1(t,x) \quad \text{in } D_2.$$

By (2), (6) and (10) we get

$$V_1(x,y) = F_2(x) + G_2(y) \leq V(t,x,y) \leq F_2(x) + G_2(y) + P_2(x) = V_2(x,y)$$

in D_2, where $V_i(x,y) > 0$ for every $(x,y) \neq (0,0)$ and $V_i(0,0) = 0$, $i = 1,2$. This means that the function $V(t,x,y)$ is positively definite and by Definition 4 it is uniformly small. Differentiate the function $V(t) = V(t,x(t),y(t))$ with respect to t, using (3), (6) and (11) we obtain

$$V'(t) \leq \left(xy - f_3(x,y)\frac{y^2}{g_2(y)} \right) f_2(x) = -W_3(x,y) \leq 0 \quad \text{in } D_2,$$

where $W_3(x,y) = \left(f_3(x,y)\frac{y^2}{g_2(y)} - xy \right) f_2(x)$. This means that the function $V'(t,x,y)$ is negatively definite, hence by Proposition 2 the proof is finished.

THEOREM 2. *Suppose that the assumptions* (2), (4)-(6), (9), (10) *and the following conditions are satisfied:*

$$(12) \qquad g_2 \in C^1(R_1) \quad \text{and} \quad g_2'(y)\text{sgn}y \geq 0 \quad \text{for } y \in R_1;$$

(13) *There exists a positive function* $f_5 \in C(R_2)$ *such that*

$$f_1(t,x,y) \geq f_5(x,y) \quad \text{in } D_2;$$

(14) *There exists nonnegative functions* $r \in C(I)$ *such that*

$$|e(t,x,y)| \leq \frac{1}{2}|y|r(t) \quad \text{and} \quad \int_0^\infty r(t)dt = R_0 < \infty.$$

Then the solutions of (1) *are uniformly bounded for* $t \geq 0$.

PROOF. For any solution $(x(t), y(t))$ of (1) and a positive constant K we define

$$V(t,x,y) = e^{-R(t)}\left(G_1(t,x) + G_2(y) + K\right) \quad \text{in } D_2,$$

where $R(t) = \int_0^t r(s)ds$. By (2), (10) and (14) we get

$$0 < W_1(x,y) = e^{-R_0}\left(P_1(x) + G_2(y) + K\right) \leq V(t,x,y) \leq$$

$$\leq P_2(x) + G_2(y) + K = W_2(x,y) \quad \text{in } D_2$$

and by (4) and (10), $W_1(x,y) \to \infty$ for $|x| \to \infty$, $|y| \to \infty$. Differentiate $V(t) = V(t,x(t),y(t))$ with respect to t. Using (5), (6), (9), (13) and (14) we obtain

$$V'(t) \leq -r(t)V(t) + \frac{y^2}{g_2(y)}\left(\frac{1}{2}r(t) - f_5(x,y)f_2(x)\right)e^{-R(t)}. \tag{15}$$

Since

$$G_2(y) = \frac{y^2}{2g_2(y)} + \frac{1}{2}\int_0^y \frac{s^2 g_2'(s)}{g_2^2(s)}ds$$

for $y \in R_1$, by (2), (6), (12)-(14) and (15) we get

$$V'(t) \leq -e^{-R_0}\frac{y^2}{g_2(y)}f_5(x,y)f_2(x) \leq 0 \quad \text{in } D_2. \tag{16}$$

We have proved that all conditions of Proposition 1 are fulfilled, therefore the solutions of (1) are uniformly bounded for $t \geq 0$. This completes the proof.

COROLLARY 2. *Let the assumptions of Theorem 2 be fulfilled. Then the trivial solution of (1) is uniformly asymptotically stable.*

PROOF. From (2), (5) and (14) it follows $g_1(t,0)=0$, $h(t,x,0)=0$ and $e(t,x,0)=0$ fo $t\in I$ and $x\in R_1$. This means that (1) has the trivial solution. For an arbitrary solution $(x(t),y(t))$ of (1) we define

$$V(t,x,y)=e^{-R(t)}\left(G_1(t,x)+G_2(y)\right) \quad \text{in } D_2,$$

where $R(t)=\int\limits_0^t r(s)ds$. By (2), (10) and (14) we get

$$V_1(x,y)=e^{-R_0}\left(P_1(x)+G_2(y)\right)\le V(t,x,y)\le P_2(x)+G_2(y)=V_2(x,y)$$

in D_2, where $V_i(0,0)=0$ and $V_i(x,y)>0$ for $(x,y)\neq(0,0)$, $i=1,2$. This means that the function $V(t,x,y)$ is positively definite and by Definition 4 it is uniformly small. Differentiating the function $V(t)=V(t,x(t),y(t))$ with respect to t, analogously as in the proof of Theorem 2 we get

$$V'(t)\le -e^{-R_0}\frac{y^2}{g_2(y)}f_5(x,y)f_2(x)=-W_3(x,y)\le 0 \quad \text{in } D_2.$$

With respect to Proposition 2 the proof is complete.

THEOREM 3. *Let the conditions of Theorem 1 be fulfilled. Moreover, suppose that the following conditions hold:*

(17) *For any positive constant c there exist a positive constant K_c such that $|g_1(t,x_1)-g_1(t,x_2)|\le K_c|x_1-x_2|$ for $|x_i|\le c$, $i=1,2$ and $t\in I$ and $|p_2(x)|\le K_c$, $f_1(t,x,y)\le K_c$ for $|x|\le c$, $|y|\le c$ and $t\in I$;*

(18) *For any positive constant c and $x,y\in C(I)$*
$$\int\limits_t^{t+1} h(s,x(s),y(s))ds\to 0,\quad t\to\infty,$$
where $|x(t)|\le c$ and $|y(t)|\le c$.

Then for all solutions $(x(t),y(t))$ of (1), $(x(t),y(t))\to(0,0)$ for $t\to\infty$.

PROOF. By Theorem 2 every solution $(x(t),y(t))$ of (1) is bounded in I, i.e. there exists a positive constant c such that $|x(t)|\le c$ and $|y(t)|\le c$ for $t\in I$. Therefore the set

$$A=\{(x(t),y(t)):\ |x(t)|\le c,\ |y(t)|\le c,\ t\in I\}\subset R_2$$

is compact. By (6), (10), (14), (17) and (18) the right sides of (1) are bounded in $I \times A$. Hence, by (16) and Proposition 3 we have

$$(x(t), y(t)) \to L = \{(x, y): \ |x| \le c, \ |y| \le c, \ \frac{y^2}{g_2(y)} f_2(x) f_5(x, y) = 0\}$$

for $t \to \infty$, i.e. the ω-limit set $\Omega(x(t), y(t))$ is a subset of L. From (6) and (13) it follows

$$L = \{(x, y): \ |x| \le c, \ y = 0\},$$

i.e.

(19) $$y(t) \to 0 \quad \text{for } t \to \infty.$$

Further we are going to prove that $\Omega(x(t), y(t)) = \{(0, 0)\}$.

Let $(a, 0) \in \Omega(x(t), y(t))$. Then by Definition 2 there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to \infty$, $n \to \infty$ and

(20) $$x(t_n) \to a, \ y(t_n) \to 0 \quad \text{for } n \to \infty.$$

Now, it suffices to show that $a = 0$. Let $a \neq 0$. Integrating the second equation of (1) from t_n to $t_n + 1$, we get

$$y(t_n + 1) - y(t_n) = -\int_{t_n}^{t_n+1} f_1(s, x(s), y(s)) f_2(x(s)) y(s) ds -$$

$$-g_1(t_n, x(t_n)) g_2(y(t_n)) - \int_{t_n}^{t_n+1} (g_1(s, x(s)) - g_1(t_n, x(t_n))) g_2(y(s)) ds -$$

$$-\int_{t_n}^{t_n+1} (g_2(y(s)) - g_2(y(t_n))) g_1(t_n, x(t_n)) ds - -\int_{t_n}^{t_n+1} h(s, x(s), y(s)) ds -$$

$$-\int_{t_n}^{t_n+1} e(s, x(s), y(s)) ds,$$

i.e.

$$g_1(t_n, x(t_n)) g_2(y(t_n)) = y(t_n) - y(t_n + 1) -$$

$$-\int_{t_n}^{t_n+1} f_1(s, x(s), y(s)) f_2(x(s)) y(s) ds -$$

$$-\int_{t_n}^{t_n+1}(g_1(s,x(s))-g_1(t_n,x(t_n)))g_2(y(s))ds-$$

$$-\int_{t_n}^{t_n+1}(g_2(y(s))-g_2(y(t_n)))g_1(t_n,x(t_n))ds-\int_{t_n}^{t_n+1}h(s,x(s),y(s))ds-$$

$$-\int_{t_n}^{t_n+1}e(s,x(s),y(s))ds.$$

Further we have

$$\begin{aligned}|g_1(t_n,x(t_n))|g_2(y(t_n))\le |y(t_n)|+|y(t_n+1)|+\\ +\int_{t_n}^{t_n+1}|f_1(s,x(s),y(s))||y(s)|f_2(x(s))ds+\\ +\int_{t_n}^{t_n+1}|g_1(s,x(s))-g_1(t_n,x(t_n))|g_2(y(s))ds+\\ +\int_{t_n}^{t_n+1}|g_2(y(s))-g_2(y(t_n))||g_1(t_n,x(t_n))|ds+\\ +\int_{t_n}^{t_n+1}|h(s,x(s),y(s))|ds+\int_{t_n}^{t_n+1}|e(s,x(s),y(s))|ds.\end{aligned} \tag{21}$$

By the well-known Lagrange's theorem there exists $\xi\in(t_n,s)$ and by (12) a positive constant a_1 such that

$$|g_2(y(s))-g_2(y(t_n))|=|y(s)-y(t_n)||g_2'(y(\xi))|\le(|y(s)|+|y(t_n)|)a_1.$$

Integrating the first equation of (1) from t_n to t we get

$$x(t)-x(t_n)=\int_{t_n}^{t}y(s)ds$$

and for $t \in [t_n, t_n+1]$ we have

$$|x(t) - x(t_n)| \le \sup_{t\in[t_n,\infty)} |y(t)| = M.$$

Let K_s be the maximum of all constants in this proof. By (14), (17), (21) and the above mentioned estimates, we obtain

$$|g_1(t_n, x(t_n))| g_2(y(t_n)) \le 2M \left(K_s^2 + K_s|g_1(t_n, x(t_n))| + 1\right) +$$

$$+ \int_{t_n}^{t_n+1} |h(s, x(s), y(s))| ds + \frac{M}{2} \int_{t_n}^{t_n+1} r(s) ds.$$

By (2), (6) and (10) the last inequality gives

$$|p_1(x(t_n))| g_2(y(t_n)) \le 2M \left(K_s^2 + K_s|g_1(t_n, x(t_n))| + 1\right) +$$

$$+ \int_{t_n}^{t_n+1} |h(s, x(s), y(s))| ds + \frac{M}{2} \int_{t_n}^{t_n+1} r(s) ds.$$

In view of (10), (14), (18) and (19), by the last inequality we have

$$p_1(x(t_n)) \to 0 \quad \text{for } n \to \infty.$$

Since $p_1 \in C(R_1)$, by (22) $p_1(a) = 0$ for $a \ne 0$. This contradicts (10) hence $x(t) \to 0$ for $t \to \infty$. The theorem is proved.

COROLLARY 3. *Let the conditions of Theorem 3 be fulfilled. Then the trivial solution of* (1) *is globally asymptotically stable.*

PROOF. By Corollary 2 the trivial solution of (1) is uniformly asymptotically stable and by Theorem 3 $(x(t), y(t)) \to (0, 0)$ for $t \to \infty$, where $(x(t), y(t))$ is an arbitrary solution of (1). Therefore by Definition 5 the trivial solution of (1) is globally asymptotically stable. The proof is finished.

THEOREM 4. *Suppose that the assumption* (2), (4)-(10) *and* (13) *are satisfied. Then the solutions of* (1) *are uniformly bounded for* $t \ge 0$.

PROOF. For any solution $(x(t), y(t))$ of (1) and a positive constant K we define

$$V(t, x, y) = G_1(t, x) + G_2(y) + K \quad \text{in } D_2.$$

By (2), (6) and (10) we get

$$0 < W_1(x,y) = P_1(x) + G_2(y) + K \leq V(t,x,y) \leq$$
$$\leq P_2(x) + G_2(y) + K = W_2(x,y)$$

in D_2 and by (4) and (10) $W_1(x,y) \to \infty$ for $|x| \to \infty$, $|y| \to \infty$. Differentiate $V(t) = V(t, x(t), y(t))$ with respect to t, using (5)-(9) and (13) we obtain

$$V'(t) \leq -\frac{y^2}{g_2(y)} f_5(x,y) f_2(x) = -W_3(x,y) \quad \text{in } D_2. \tag{22}$$

With respect to Proposition 1. the proof is complete.

COROLLARY 4. *Let the conditions of Theorem 4 be fulfilled. Then the trivial solution of* (1) *is uniformly asymptotically stable.*

PROOF. From (2), (5) and (8) it follows $g_1(t,0) = 0$, $h(t,x,0) = 0$ and $e(t,x,0) = 0$ for $t \in I$ and $x \in R_1$. This means that (1) has the trivial solution. For an arbitrary solution $(x(t), y(t))$ of (1) we define

$$V(t,x,y) = G_1(t,x) + G_2(y) \quad \text{in } D_2.$$

By (2), (6) and (10) we have

$$V_1(x,y) = P_1(x) + G_2(y) \leq V(t,x,y) \leq P_2(x) + G_2(y) = V_2(x,y)$$

in D_2, where $V_i(0,0) = 0$ and $V_i(x,y) > 0$ for $(x,y) \neq (0,0)$, $i = 1,2$. This means that the function $V(t,x,y)$ is positively definite and by Definition 4 it is uniformly small. Similarly as in the proof of Theorem 4 we get (22). With respect to Proposition 2 the proof is complete.

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