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APPROXIMATION OF FUNCTIONS BY RIESZ, (C,α) AND TYPICAL MEANS FOURIER SERIES OF THESE FUNCTIONS

Yuri L. NOSENKO

Summary. The representation of the deviation of integrable - functions and Riesz, (C,α) and typical means for Fourier series of these functions are obtained. The remainders are estimated by appropriative moduli of smootheness of given functions from both above and below.

Let a function $f\in L_p$, $1\le p\le\infty$, 2π-periodic and

$$f(x)-\sum_{k=-\infty}^{\infty} C_k e^{ikx}=\sum_{k=-\infty}^{\infty} A_k(x) - \tag{1}$$

$$R_n(f;x)=\sum_{|k|\le n}\left(1-\frac{|k|}{n+1}\right)A_k(x)\,,$$

$$\sigma_n^{\alpha}(f;x)=\sum_{|k|\le n}\frac{A_{n-k}}{A_n^{\alpha}}A_k(x)\,,\; A_n^{\alpha}=(\alpha+1)(\alpha+2)\dots(\alpha+n)/n!$$

i.e. Riesz, (C,α) and typical ones $\alpha=1$ in both situations, namely $\sigma_n(f;x)=\sigma_n^{-1}(f;x)=R_n^1(f;x)$.

The deviations $f(x)-\sigma_n^{\alpha}(f;x)$ and $f(x)-R_n^{\alpha}(f;x)$ were investigated by some authors. The main terms of these deviations were obtained and the remaining ones were estimated from above in the terms of moduli of smoothness of f.
(See [1]-[3] for reference and further comments).

Here is one of these results to M.M.Lekishvili [1]: there exists $C=C(\lambda,\rho)>0$ such that for any $\lambda>0$ and f,α as above

$$f(x)-\sigma_n^{\alpha}(f;x) = -\frac{\alpha}{2\pi}\int_{\lambda}^{\infty}\Delta^2_{\frac{t}{n}} f(x)\, t^{-2}dt + \tau_n(x)$$

$$\|\tau_n(x)\|_p \le C\omega_2\left(f;\frac{1}{n+1}\right)_p .$$

In the case of Riesz means, for analogous result see [2].

We have obtained some new results in this direction.

Theorem 1. *For $f\in L_p$, $1\le p\le\infty$, $\alpha>0$ and any $\lambda>0$ there exist $C_i=C_i(\lambda,p)$, $i=1,2$, such that*

$$f(x)-\sigma_n^{\alpha}(f;x) = \alpha(f-\sigma_n(f;x))+\tau_n(f;x) ,$$

$$f(x)-R_n^{\alpha}(f;x) = \alpha(f-\sigma_n(f;x))+\tau_n(f;x) ,$$

$$C_1\omega_2\left(f;\frac{1}{n+1}\right)_p \le \|\tau_n(f;x)\|_p \le C_2\omega_2\left(f;\frac{1}{n+1}\right)_p . \qquad (2)$$

Theorem 2. *For f,p,α,λ as in theorem 1 there exist $C_i=C_i(\lambda,p)>0$, $i=1,2$ such that*

$$f(x)-\tilde{\delta}_n^{\alpha}(f;x) = -\frac{\alpha}{2\pi}\int_{\lambda}^{\infty}\tilde{\Delta}^2_{\frac{t}{n}} f(x)\, t^{-2}dt+\tau_n(f;x) ,$$

$$f(x)-R_n^{\alpha}(f;x) = -\frac{\alpha}{2\pi}\int_{\lambda}^{\infty}\tilde{\Delta}^2_{\frac{t}{n}} f(x)\, t^{-2}dt+\tau_n(f;x) ,$$

$$C_1\omega_2\left(f;\frac{1}{n+1}\right)_p \le \|\tau_n(f;x)\|_p \le C_2\omega_2\left(f;\frac{1}{n+1}\right)_p . \qquad (3)$$

Theorem 3. *For f,p,λ as in theorem 1 and $\tilde{f}$ being conjugated to f there exist $C_i=C_i(\lambda,p)>0$, $i=1,2$ and the absolute constants $C_i^{*}>0$, $i=1,2,3$ such that*

$$f(x)-\sigma_n(f;x)=C_1^*\int_\lambda^\infty \mathring{\Delta}^2_{\frac{t}{2n}} f(x)\,t^{-2}dt+C_2^*\int_\lambda^\infty \mathring{\Delta}^3_{\frac{t}{2n}} \tilde{f}(x)\,t^{-3}dt+$$

$$+C_3^*\int_\lambda^\infty \mathring{\Delta}^4_{\frac{t}{2n}} f(x)\,t^{-4}dt+\tau_n(f;x)\,,$$

$$C_1\omega_4\left(f;\frac{1}{n+1}\right)_p\le\|\tau_n(f;x)\|_p\le C_2\omega_4\left(f;\frac{1}{n+1}\right)_p\,.$$

The constants C_i (or C_i^*) may be of course different in different occurrences.

Some commentaries.

Theorem 1 shows that it is possible to obtain the representations for $f-\sigma_n^\alpha(f)\left(f-R_n^\alpha(f)\right)$ immediately when having the reprezentation for $f-\sigma_n(f)$ only without a special proof.

Theorem 2 gives the exact order for the remainder $\tau_n(f;x)$ that is $\tau_n(f;x)$ is estimated by moduli of smoothness of f not only from above as in papers of other authors ([1)], [2]) but from below too (as in [3]). Theorem 3 represents further developments of above mentioned results in the case of typical means and the moduli of smoothness of higher orders of f (with higher accuracy).

The proofs of the theorems 1-3 are based on the principle of Fourier series proposed by R.M.Trigub [4] and on the appropriative theorems on the multiplicators.

Some details. To prove (2) we'll prove at first that

$$C_1\|f-\tau_n^2(f)\|_p\le\|\tau_n(f)\|_p\le C_2\|f-\tau_n^2(f)\|_p\,,$$

$$\tau_n^\alpha(f;x)=\sum_{|k|\le n}\left(-\left(\frac{|k|}{n+1}\right)^\alpha\right)A_k(x)\quad\text{- Riesz means.}$$

But the exact order of deviation $f(x)-\tau_n^2(f;x)$ is well-known [4]. To prove for example the right-side inequality in (2) we will construct transitional function $(x=|k|/(n+1))$

$$\Lambda(x)=\begin{cases}(1-\alpha x-(1-x)^{\alpha})x^{-2}, & 0<x\le 1\\ 1-\alpha, & x\ge 1,\end{cases}\qquad \Lambda(0)=\frac{1}{2}\alpha(1-\alpha).$$

After this it remains to use the comparison principle and the theorems on the multiplicators.

In the case of (3) (right-side inequality) the transitional sequence is more complicated, namely

$$\Lambda_k=\begin{cases}\left(1-\frac{\alpha|k|}{n+1}-\frac{A_n^{\alpha}-|k|}{A_n^{\alpha}}\right)\frac{(n+1)^2}{k^2}, & |k|\le n\\ 1-\alpha; & |k|>n,\end{cases}$$

$$\Lambda_0=\alpha(1-\alpha).$$

As for the other cases in theorems 1-3, the situation appears like to the mentioned one.

REFERENCES

1. LEKISHVILI, M.M., Concerning the approximations of periodic functions by (C,α) means, Mathem.Sborn.-1983-121 (163); 4(8), pp.499-509 (in Russian)
2. FALALEEV, L.P., Concerning the matrix methods of summubility in L_p, Math.Zam.-1993-55, n5, pp.111-118 (in Russian)
3. NOSENKO, Y.L., Approximation of Functions by Riesz means of their Fourier series, Harmonic analysis and developments of approximative methods, Kiev, Inst.of Math.-1989, pp.83-84 (in Russian)
4. TRIGUB, R.M, Absolute convergence of Fourier integrals and approximation of functions of polynomials on thorus, Izv.AN SSSR, mat.-1980-44, n6, pp.1379-1409 (in Russian).

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Donetsk Technical University
Donetsk, Ukraine