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ON SOME EXTENSIONS OF COLLINEATIONS OF A DESARGUESIAN PROJECTIVE PLANE

Angela VASIU, Nicolae OPREA

SUMMARY. In [7] and [8] was introduced the notion of anchor in a translation projective plane, by which can be realised the extension of collineations defined on a subset of a translation projective plane.

Let Π and Π' be two translation projective planes and $\mathcal{P}$, $\mathcal{P}'$, their sets of points, d_∞, d'_∞ their impropre lines and T, T' the translation groups of the given planes.

For a subset $C\subset\mathcal{P}$ we denote $\Omega(C,\mathcal{P}')$ the set of all injective collineations from C to $\mathcal{P}'$ and:

$$C_* = C\setminus d_\infty \ , \ C_\tau = C_* \cap \tau^{-1}(C_*) \ , \forall\tau\in T$$

Definition 1.1. The set $C\subset\mathcal{P}$ is called an anchor, if for any translation plane π', for any $\varphi\in\omega(C,\mathcal{P}')$ and for any $\tau\in T$, there exists $\tau'\in T'$ such that:

1. $\tau'(\varphi(M)) = \varphi(\tau(M))\ ,\ \forall M\in C$

2. $(\tau_2\circ\tau_1) = \tau'_2\circ\tau'_1$ and $cen\tau_1 = cen\tau_2$

   $cen\tau'_1 = cen\tau'_2 \ \ \forall\tau_1 , \ \tau_2\in T.$

Definition 1.2. The set $c\subset\mathcal{P}$ for which is true only the first condition is called a semi-anchor.

For the anchors introduced by F.RADO in [5] he proved that any collineation of a plane defined on an anchor, can be extended to a collineation of the whole plane. In [4] are given exemples of semianchors.

In the case of desarguesian projective planes, the collineations are represented by semilinear transformations of the corresponding vector spaces, and as it results from [1], [2], these can be characterised by a dilatation group by an elation group of a plane. The role of the translation group is taken by the delatation and elation groups, in a desarguesian plane.

In [9] we generalised the notion of anchor for a desarguesian projective plane. In [10] are given for a pappusian projective plane some anchors which correspond to some special collineation groups.

2. Let Π and Π' be two desarguesian projective planes, $\wp$ and $\wp'$ their point sets, (GL), (GL)′ the groups of projective collineations of π and π'. We denote by $(GL)_d$, $(GL)_{d'}$ the group of the projectivities of a line $d\subset\pi$, $d'\subset\pi'$. Let $C\subseteq\wp$ be a set of points and $\omega(C,\wp')$ the set of injective collineations φ: $C \to \wp'$.

Definition 2.1. Let $g\in GL$ (or $g\in(GL)_d$) and $C_g = C\cap g^{-1}(C)$. We say that C is compatible with g if $C_g\neq\emptyset$, $C_g\not\subset\mathscr{F}$, where $\mathscr{F}$ is the set of fixed points of g, if for any $\varphi\in\Omega(C,\wp')$ there exists an unique colliniation $g'\in(GL)'$ (or $g'\in(GL)'_{d'}$) such that for any $M\in C_g$ to be satisfied the relation:

$$g'(\varphi(M))=\varphi(g(M)) \tag{1}$$

Definition 2.2. Let G be a subgroup of the group GL (or of the group $(GL)_d$). The set $C\subseteq\wp$ is called a G-anchor, if it is compatible with any $g\in G$ and if for any $\varphi\in\Omega(C,\wp')$ we have:

$$g_2\circ g'_1=g'_2\circ g'_1\ ,\ \forall g_1,g_2\in G \tag{2}$$

Definition 2.3. Let G_1 and G_2 be two subgroups of the group GL. A set of points $C\subset\wp$ is called a G_1-G_2-anchor if C is G_1-anchor and G_2-anchor, C is compatible with $g_2^{g_1}=g_1\circ g_2\circ g_1^{-1}$ for any $g_1\in G_1$, $g_2\in G_2$ and if:

$$(g_2^{g_1})'=(g'_2)^{g'_1} \tag{3}$$

Definition 2.4. In the desarguesian projective plane π let Q be a point and d a line, nonincident with Q. A projective collineation δ which admits Q and all points of d as fixed points, is called a dilatation with d as axis and Q as centre. The set of these dilatations forms a group, noted by $D_{d,Q}$.

Definition 2.5. If H is a point incident with d, a projective collineation $\tau: \pi \to \pi'$ by which all the points of d and all the lines which are incident with H are fixed, is called an elation with the centre in H and of axis d.

The set of elations forms a group denoted by $T_{d,H}$. The set of elations:

$T_d=\{\tau \in T_{d,H} | H \in d\}$ is also a group, called the group of elations with axis d.

We use the next theorem from [9].

In a desarguesian projective plane let Q be a point and d a line, $Q \notin d$, and $C \subset \mathcal{P}$, a set of points which contains Q and at least two points H_1 and H_2 of the line d. If C is a $D_{d,Q}$-T_d anchor, then any collineation $\varphi: C \to \mathcal{P}'$ can be extended on the whole plane π, that is, there is an unique projective collineation $f: \mathcal{P} \to \mathcal{P}'$ such that $f|_C=\varphi$.

In [10] we found three types of anchors:

1). $C = \Gamma \cup d$, where Γ is a propre conic, d a line, $\Gamma \cap d = \{A,B\}$, C is a $G_{A,B}$-anchor, when $G_{A,B}$ is the subgroup of projectivities of the group $(GL)_d$, having A,B as fixed points.

2). $C = \Gamma \cup d$, where Γ is propre conic, d a tangent to Γ in $A \in \Gamma$. C is a G_A-anchor where G_A is the subgroup of $(GL)_d$ having A the only fixed point.

3). $C = \Gamma \cup d \cup \{P\}$, Γ and d are the same as in case 1, and P belongs to tangent in A to Γ. C is a compatible with all involutions of d which have a fixed point in A.

4). In this note we give C new anchor in a projective desarguesian plane.

Theorem 3.1. *In a projective desarguesian plane let $C = d_1 \cup d_2 \cup d_3$, d_i lines and $\{O\} = d_1 \cap d_2 \cap d_3$, then C is a $T_{d_i,O}$-anchor, for $i=\overline{1,3}$.*

Proof. Let τ be an elation from $T_{d_3,O}$ and $M_1' \in d_1$, then $\tau(M_1') \in d_1 \subset C$. If $\varphi: C \to \mathcal{P}'$ is a collineation, let τ' be the elation from $T_{d_3,o}$, $d'_3=\varphi(d_3)$ and $o'=\varphi(o)$ determined by:

$$\tau'(\varphi(M_1^o))=\varphi(\tau(M'_o)) \quad (4)$$

Let M_2 be an arbitrary point on the line d_2. Then $\tau(M_2)\in d_2$ (by definition of elation τ) and we have:

$$(M_1^o+M_2)\cap[\tau(M_1^o)+\tau(M_2)]\in d_3 \quad (5)$$

$$[\varphi(M_1^o)+\varphi(M_2)]\cap[\varphi(\tau(M_1^o)+\varphi(\tau(M_2))\in d'_3 \quad (6)$$

From (4), (5) and (6) we obtain:

$$\tau'(\varphi(M_2))=\varphi(\tau(M_2)),\ \forall M_2\in d_2 \quad [7]$$

Similardly we can deduce:

$$\tau'(\varphi(M_1))=\varphi(\tau(M_1)),\ \forall M_1 d_1$$

If $M_3\in d_3$, then $\tau(M_3)=M_3$, $\tau'(\varphi(M_3))=\varphi(M_3)=\varphi(\tau(M_3))$. .

Also for any $M\in C$ we have:

$$\tau'(\varphi(M))=\varphi(\tau(M)) \quad (8)$$

We deduce that C is compatible with any elation τ from $T_{d_3,o}$.

For $\tau_1,\tau_2\in T_{d_3,o}$ and $M\in C$ the points $\tau_1(M)$ and $\tau_2\circ\tau_1(M)$ belong to C, and from (8) it results that:

$$(\tau_2\circ\tau_1)\circ\varphi(M)=\varphi(\tau_2\circ\tau_1(M)) \quad \text{and:}$$

$$\varphi(\tau_2\circ\tau_1)(M)=\varphi[\tau_2(\tau_1(M))]=\tau'_2(\varphi(\tau_1(M))=\tau'_2(\tau'_1(M))$$

and thus we have:

$$(\tau_2\circ\tau_1)=\tau'_2\circ\tau'_1$$

that is C is a $T_{d_{1,o}}$ -anchor.

Analogously we can prove the C is also a $T_{d_{1,o}}$ and $T_{d_{2,o}}$ anchor, what proves the theorem.

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UNIVERSITY OF BAIA MARE
DEPARTMENT OF MATHEMATICS
4800 BAIA MARE
ROMANIA