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A POST TYPE THEOREM FOR *(m,n)* RINGS WITH UNIT AS A SYSTEM OF *(n-1)* ELEMENTS

Maria S.POP and Lăcrimioara IANCU

Abstract: In this paper a binary reduct of an (m,n) ring is constructed, an usual ring on a covering set of an (m,n) ring is constructed and an isomorphism between these two rings is determined. As main result the analogous for (m,n) rings of the Post Coset Theorem for n-groups is given.

1. INTRODUCTION

The study of (m,n) -rings has provided the subject matter for several publications.

Dörnte,[4], considered a generalized group in 1928; he studied systems with one m-ary operation subject to associativity laws and to the existence of solutions to equations.

Post,[7],called this algebras polyadic groups and examined their structure in 1940; in an important result, generally referred to as the Post Coset Theorem, he showed that an m-group is a coset of an invariant subgroup, called the associated group,in an ordinary 2-group, called the covering group, and that the corresponding factor group is cyclic of order m-1. The m-ary operation in the m-group is the operation of the cover restricted to products involving admissible numbers of terms from the coset.

Boccioni,[1], established an analogous of the Post Coset Theorem for (n,2)-rings: an (n,2) ring A is a coset of an ideal I of a ring

R and R/I is isomorphic to the ring of integers modulo $n-1$.
Leeson and Butson,[5], showed that an (m,n) ring A is a coset z+I of an ideal I of a (2,n) ring R with $\left(z^{(m)}\right)_{+} \in z+I$ and (R/I,+) is isomorphic to $(\mathbb{Z}_{m-1},+)$; conversely any such coset is an (m,n) ring.

The (2,n) ring R was called the Post cover of A and the ideal I was called the associated (2,n) ring.
Crombez,[3], as well, established a Post coset theorem for (n,m) rings.

2. NOTATION AND PRELIMINARY RESULTS

Definition 2.1 An n-semigroup is an algebraic system $(A,()_\circ)$ with one n-ary operation $\circ: A^n \to A$, $n \in N$, $n \geq 2$, so that for any set of elements $a_1, a_2, \ldots, a_{2n-1} \in A$, and any $k=1,\ldots,n-1$ it is true that $((a_1,\ldots,a_n)_\circ, a_{n+1},\ldots,a_{2n-1})_\circ =$

$= (a_1,\ldots,a_k,(a_{k+1},\ldots,a_{k+n})_\circ, a_{k+n+1},\ldots,a_{2n-1})_\circ$,

shortly $\left((a_1^n)_\circ, a_{n+1}^{2n-1}\right)_\circ = \left(a_1^k, (a_{k+1}^{k+n})_\circ, a_{k+n+1}^{2n-1}\right)_\circ$

Definition 2.2 An n - group $(A,()_\circ)$ is an n - semigroup $(A,()_\circ)$ in which the equations $(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n)_\circ = a_i$ have a unique solution in A for arbitrary $a_1,\ldots,a_n \in A$ and for each $i \in \{1,\ldots,n\}$.

An n - semigroup (n - group) $(A,()_\circ$ is called:
commutative if the operation "O" is invariant under each permutation of the elements involved

semicommutative if $(a_1, a_2, \ldots, a_{n-1}, a_n)_* = (a_n, a_2, \ldots, a_{n-1}, a_1)_*$ for any set of elements $a_1, \ldots, a_n \in A$.

entropic (or medial) if

$$((a_{11}, a_{12}, \ldots, a_{1n})_*, (a_{21}, a_{22}, \ldots, a_{2n})_*, \ldots, (a_{n1}, a_{n2}, \ldots, a_{nn})_*)_* =$$

$$= ((a_{11}, a_{21}, \ldots, a_{n1})_*, (a_{12}, a_{22}, \ldots, a_{n2})_*, \ldots, (a_{1n}, a_{2n}, \ldots, a_{nn})_*)_*,$$

for any $a_{ij} \in A$, $i, j \in \{1, \ldots, n\}$.

An element $a \in A$ of an n - semigroup $(A, ()_*)$ is called **idempotent** if $\left(\overset{(n)}{a}\right)_* = a$.

In an n - group $(A, ()_*)$ the unique solution of the equation $(a, a, \ldots, a, x)_* = a$ is called the **querelement** of **a** and is denoted by $\overline{a}$.

The $(n-1)$-ad $u_1, \ldots, u_{n-1} \in A$ is a **right unit** as a system of $(n-1)$ elements if $(a, u_1, \ldots, u_{n-1})_* = a$, for each $a \in A$. In an n - group $\overset{(i)}{b}\ \overline{b}\ \overset{(n-i-2)}{b}$ is a unit for each $b \in A$ and for each $i = 0, 1, \ldots, n-2$.

Proposition 2.3 *(Post [7] Coset Theorem for n - groups) An n-group A is a coset $x+N$ of a normal subgroup N called the associated 2 - group, in a 2 - group G, which A generates, called the covering group G, with $(G/N, +) \cong (\mathbb{Z}_{n-1}, +)$. Conversely, such a coset (for some G,N and x) is an n - group. Furthemore, G is an abelian 2 - group if and only if A is an abelian n - group.*

Definition 2.4 An algebra $(R, [\,], ()_*)$ is an **(m,n) ring** if:

(i) $(R, [\,])$ is a commutative m - group

(ii) $(R, ()_*)$ is an n - semigroup, and

(iii) the following distributive law hold for all choices of $a_1, \ldots, a_n, b_1, \ldots, b_m \in R$ and for all choices of $i \in \{1, 2, \ldots, n\}$:

$$(a_1, \ldots, a_{i-1}, [b_1, b_2, \ldots, b_m], a_{i+1}, \ldots, a_n)_\cdot =$$
$$= [(a_1, \ldots, a_{i-1}, b_1, a_{i+1}, \ldots, a_n)_\cdot,$$
$$(a_1, \ldots, a_{i-1}, b_2, a_{i+1}, \ldots, a_n)_\cdot, \ldots, (a_1, \ldots, a_{i-1}, b_m, a_{i+1}, \ldots, a_n)_\cdot]$$

An element $b \in R$ is an additive **idempotent** if $[b, b, \ldots, b] = b$ and a multiplicative **idempotent** if $(b, b, \ldots, b)_\cdot = b$. If both of these conditions are satisfied, b will be called an **idempotent** of R.

The element $\bar{b}$ will denote the additive querelement of b.

It is known that the additive querelement has the following properties in an (m,n) ring: $\overline{[b_1, \ldots, b_m]} = [\bar{b}_1, \ldots, \bar{b}_m]$ and

$$\overline{(b_1, \ldots, b_n)_\cdot} = (b_1, \ldots, b_{i-1}, \bar{b}_i, b_{i+1}, \ldots, b_n)_\cdot \quad \text{for } i = 1, 2, \ldots, n.$$

If the multiplicative n - semigroup contains an n - group, the element $\underline{b}$ will denote the multiplicative querelement of b therein.

An element 0 in an (m,n) ring R is called a **zero** of R if

$$0 = (0, r_2, \ldots, r_n)_\cdot = \ldots = (r_1, r_2, \ldots, r_{n-1}, 0)_\cdot \quad \text{for each choice of } r_i \in R.$$

An (m,n) ring may have at most one zero.

A zero of R is an idempotent of R.

3.

Let $(R, [\ ], (\ )_\cdot)$ be an (m,n) ring and $a, u_1, \ldots, u_{n-2} \in R$ fixed elements of R.

Define two binary operations on R:

$$\oplus : R \times R \to R, \qquad x \oplus y = [x, \overset{(m-3)}{a}, \bar{a}, y] \qquad (1) \qquad \text{and} \quad \odot : R \times R \to R,$$

$$x \odot y = [(x, u_1^{n-2}, y)_\cdot, \overset{(m-3)}{(x, u_1^{n-2}, a)_\cdot}, (x, u_1^{n-2}, \bar{a})_\cdot, \overset{(m-3)}{(a, u_1^{n-2}, y)_\cdot},$$

$$(\bar{a}, u_1^{n-2}, y)_\cdot, (a, u_1^{n-2}, a)_\cdot, a] \qquad (2)$$

Proposition 3.1 *If* $(R,[\,],(\,))$, *is an* (m,n) *ring, then* $(R,\oplus,\odot)$ *is a ring called the binary reduced ring with respect to the elements* $a,u_1,\ldots,u_{n-2}\in R$ *and denoted by* $red_a^{+}(red_{u_1^{n-2}}^{\cdot}(R,[\,],(\,),))$.

Proof. It is easily verified that $(R,\oplus)$ is a (binary) abelian group (in which the identity element is a and $-x=[a,\overset{(m-3)}{x},\overline{x},a]$)

We shall prove the associative law and the distributive laws for the operation $\odot$.

$(x\odot y)\odot z=[a,(a,u_1^{n-2},a),(x,\overset{(m-3)}{u_1^{n-2}},a),(x,u_1^{n-2},\overline{a}),$

$(a,\overset{(m-3)}{u_1^{n-2}},y),(\overline{a},u_1^{n-2},y),(x,u_1^{n-2},y),]\odot z=$

$=[(x,u_1^{n-2},y,u_1^{n-2},z),(a,u_1^{n-2},a,u_1^{n-2},z),(x,\overset{(m-3)}{u_1^{n-2}}a,u_1^{n-2},z),$

$(\overline{x},u_1^{n-2},a,u_1^{n-2},z),(a,\overset{(m-3)}{u_1^{n-2}},y,u_1^{n-2},z),(\overline{a},u_1^{n-2},y,u_1^{n-2},z),$

$(a,u_1^{n-2},z),(a,u_1^{n-2},a),(x,\overset{(m-3)}{u_1^{n-2}},y,u_1^{n-2},a),a,(a,\overset{(m-3)}{u_1^{n-2}},a,u_1^{n-2},a),$

$(a,\overset{(m-3)}{u_1^{n-2}},a),(x,u_1^{n-2},\overset{(m-3)^2}{a},u_1^{n-2},a),(x,u_1^{n-2},\overset{(m-3)}{\overline{a}},u_1^{n-2},a),$

$(a,u_1^{n-2}\overset{(m-3)^2}{y},u_1^{n-2},a),(\overline{a},u_1^{n-2},\overset{(m-3)}{y},u_1^{n-2},a),(x,u_1^{n-2},y,u_1^{n-2},\overline{a}),$

$(a,u_1^{n-2},a,u_1^{n-2},\overline{a}),(a,u_1^{n-2},\overline{a}),(x,u_1^{n-2},\overset{(m-3)}{a},u_1^{n-2},\overline{a}),$

$(x,u_1^{n-2},\overline{\overline{a}},u_1^{n-2},\overline{a}),(a,u_1^{n-2},\overset{(m-3)}{y},u_1^{n-2},\overline{a}),$

$$(\overline{a}, u_1^{n-2}, y, u_1^{n-2}, \overline{a})_{\cdot}, (a, \overset{(m-3)}{u_1^{n-2}}, z)_{\cdot}, (\overline{a}, u_1^{n-2}, z)_{\cdot}] =$$

$$= [(x, u_1^{n-2}, y, u_1^{n-2}, z)_{\cdot}, (x, u_1^{n-2}, a, u_1^{n-2}, a)_{\cdot}, (a, u_1^{n-2}, y, u_1^{n-2}, a)_{\cdot},$$

$$(a, u_1^{n-2}, a, u_1^{n-2}, z)_{\cdot}, (x, u_1^{n-2}, \overset{(m-3)}{a}, u_1^{n-2}, z)_{\cdot}, (x, u_1^{n-2}, \overline{a}, u_1^{n-2}, z)_{\cdot},$$

$$(x, u_1^{n-2}, \overset{(m-3)}{y}, u_1^{n-2}, a)_{\cdot}, (x, u_1^{n-2}, y, u_1^{n-2}, \overline{a})_{\cdot}, (a, u_1^{n-2}, \overset{(m-3)}{y}, u_1^{n-2}, z)_{\cdot},$$

$$(\overline{a}, u_1^{n-2}, y, u_1^{n-2}, z)_{\cdot}, a, (a, u_1^{n-2}, \overset{(m-3)}{a}, u_1^{n-2}, a)_{\cdot}, (\overline{a}, u_1^{n-2}, a, u_1^{n-2}, a)_{\cdot}]$$

$$x \odot (y \odot z) = x \odot [a, (a, u_1^{n-2}, a)_{\cdot}, (y, \overset{(m-3)}{u_1^{n-2}}, a)_{\cdot}, (y, u_1^{n-2}, \overline{a})_{\cdot},$$

$$(a, \overset{(m-3)}{u_1^{n-2}}, z)_{\cdot}, (\overline{a}, u_1^{n-2}, z)_{\cdot}, (y, u_1^{n-2}, z)_{\cdot}] =$$

$$= [(x, u_1^{n-2}, y, u_1^{n-2}, z)_{\cdot}, (x, u_1^{n-2}, a, u_1^{n-2}, a)_{\cdot}, (x, u_1^{n-2}, a)_{\cdot},$$

$$(x, u_1^{n-2}, \overset{(m-3)}{y}, u_1^{n-2}, a)_{\cdot}, (x, u_1^{n-2}, y, u_1^{n-2}, \overline{a})_{\cdot}, (x, u_1^{n-2}, \overset{(m-3)}{a}, u_1^{n-2}, z)_{\cdot},$$

$$(x, u_1^{n-2}, \overline{a}, u_1^{n-2}, z)_{\cdot}, (a, u_1^{n-2}, a)_{\cdot}, a, (x, \overset{(m-3)}{u_1^{n-2}}, a)_{\cdot}, (x, u_1^{n-2}, \overline{a})_{\cdot},$$

$$(a, u_1^{n-2}, \overset{(m-3)}{y}, u_1^{n-2}, z)_{\cdot}, (\overline{a}, u_1^{n-2}, y, u_1^{n-2}, z)_{\cdot}, (a, u_1^{n-2}, \overset{(m-3)}{a}, u_1^{n-2}, a)_{\cdot},$$

$$(a, \overset{(m-3)}{u_1^{n-2}}, a)_{\cdot}, (a, u_1^{n-2}, \overset{(m-3)^2}{y}, u_1^{n-2}, a), (a, u_1^{n-2} \overset{(m-3)}{y}, u_1^{n-2}, \overline{a})_{\cdot},$$

$$(a, u_1^{n-2}, \overset{(m-3)^2}{a}, u_1^{n-2}, z)_{\cdot}, (\overline{a}, u_1^{n-2}, \overset{(m-3)}{a}, u_1^{n-2}, z), (\overline{a}, u_1^{n-2}, a, u_1^{n-2}, a)_{\cdot},$$

$$(\overline{a}, u_1^{n-2}, a)_{\cdot}, (\overline{a}, u_1^{n-2}, \overset{(m-3)}{y}, u_1^{n-2}, a)_{\cdot}, (\overline{a}, u_1^{n-2}, y, u_1^{n-2}, \overline{a})_{\cdot},$$

$$(\overline{a}, u_1^{n-2}, \overset{(m-3)}{a}, u_1^{n-2}, z)_{\cdot}, (\overline{a}, u_1^{n-2}, \overline{a}, u_1^{n-2}, z)_{\cdot}] =$$ after some calculus

$$= (x \odot y) \odot z$$

$$x\odot(y\oplus z)=x\odot[y,\overset{(m-3)}{a},\bar{a},z]=[a,(a,u_1^{n-2},a)_\cdot,(x,\overset{(m-3)}{u_1^{n-2}},a)_\cdot,$$

$$(x,u_1^{n-2},\bar{a})_\cdot,(a,\overset{(m-3)}{u_1^{n-2}},y),(a,\overset{(m-3)^2}{u_1^{n-2}},a)_\cdot,$$

$$(a,\overset{(m-3)}{u_1^{n-2}},\bar{a}),(a,\overset{(m-3)}{u_1^{n-2}},z)_\cdot,(\bar{a},u_1^{n-2},y)_\cdot,(\bar{a},\overset{(m-3)}{u_1^{n-2}},a)_\cdot,(\bar{a},u_1^{n-2},\bar{a})_\cdot,$$

$$(\bar{a},u_1^{n-2},z)_\cdot,(x,u_1^{n-2},y)_\cdot,(x,\overset{(m-3)}{u_1^{n-2}},a),(x,u_1^{n-2},\bar{a})_\cdot,$$

$$(x,u_1^{n-2},z)_\cdot]=[a,(a,u_1^{\overset{(2)}{n-2}},a)_\cdot,(x,u_1^{\overset{(m-1)}{n-2}},a)_\cdot,(x,u_1^{n-2},\bar{a})_\cdot,$$

$$(a,u_1^{\overset{(m-1)}{n-2}},y),(\bar{a},u_1^{n-2},y),(a,\overset{(m-3)}{u_1^{n-2}},z)_\cdot,(\bar{a},u_1^{n-2},z)_\cdot,$$

$$(x,u_1^{n-2},y)_\cdot,(x,u_1^{n-2},z)_\cdot]$$

$$x\odot y\oplus x\odot z=[a,(a,u_1^{n-2},a)_\cdot,(x,\overset{(m-3)}{u_1^{n-2}},a)_\cdot,(x,u_1^{n-2},\bar{a})_\cdot,(a,\overset{(m-3)}{u_1^{n-2}},y)_\cdot,$$

$$(\bar{a},u_1^{n-2},y),(x,u_1^{n-2},y)_\cdot,\overset{(m-3)}{a},\bar{a},a,(a,u_1^{n-2},a)_\cdot,$$

$$(x,\overset{(m-3)}{u_1^{n-2}},a),(x,u_1^{n-2},\bar{a})_\cdot,(a,\overset{(m-3)}{u_1^{n-2}},z)_\cdot,(\bar{a},u_1^{n-2},z)_\cdot,(x,u_1^{n-2},z)_\cdot]=$$

$$=x\odot(y\oplus z)$$

Remark 3.2(i) If $u_1^{n-2}a$ is a right unit and au_1^{n-2} is a left unit as a system of $(n-1)$ elements in the (m,n) ring R, then the binary operations are:

$$x\oplus y=[x,\overset{(m-3)}{a},\bar{a},y],\quad x\odot y=[a,a,\overset{(m-3)}{x},\bar{x},\overset{(m-3)}{y},\bar{y},(x,u_1^{n-2},y)_\cdot] \tag{3}$$

(ii) If a is a zero in the (m,n) ring R then the operation $\odot$ is given by: $x\odot y=(x,u_1^{(n-2)},y)$. (4)

Example 3.3 (i) Let $(\mathbb{Z}_8, [\,], (\,)_\cdot)$ be the (3.3) – ring of integers modulo 8, where the operations are: $[x_1, x_2, x_3] = x_1 + x_2 + x_3$, $(x_1, x_2, x_3)_\cdot = \hat{3}x_1 x_2 x_3$. $\hat{0}$ is a zero element in this commutative (3.3) -ring and $(\hat{1}, \hat{3})$ is a unit as a system of two elements. $red_{\hat{1}}^{+}(red_{\hat{3}}^{\cdot}(\mathbb{Z}_8, [\,], (\,)_\cdot)$ is the ring $(\mathbb{Z}_8, \oplus, \odot)$ where the operations are:

$$x \oplus y = [x, \hat{1}, y] = x + \hat{7} - y \quad \text{and} \quad x \odot y = \hat{2} + \hat{7}x + \hat{7}y + xy$$

$\hat{7}$ is a zero elements in this commutative ring and $\hat{2}$ is a neuter element for the operation $\odot$.

(ii) Let $R = \{a, b\}$ a (3,4) commutative ring, where the operations are:

$a+a+a=a$	$aaaa=a$
$a+a+b=b$	$aaab=b$
$a+b+b=a$	$aabb=a$
$b+b+b=b$	$abbb=b$
	$bbbb=a$

In this (3,4) ring a is an additive and multiplicative idempotent and b is an additive idempotent: abb and aaa are units as systems of three elements.

$red_b^{+}(red_{ab}^{\cdot}(R, +, \cdot))$ is the ring $(R, \oplus, \odot)$, where the operations are:

$x \oplus y = x + y + b$, $x \odot y = x + y + xaby$

b is a zero element in this commutative ring

(iii) Let $R = \{a, b\}$ a (3,4) commutative ring, where addition is:

$(a, a, a)_+ = b$
$(a, a, b)_+ = a$
$(a, b, b)_+ = b$
$(b, b, b)_+ = a$

and multiplication is the same as before.

In this (3,4) ring a is a multiplicative idempotent, abb and aaa are units as systems of three elements.

$red_b^*(red_{ab}^{\cdot}(R,()_+\cdot))$ is the ring $(R_1,\oplus,\odot)$, where the operations are:

$x\oplus y=x+y+a$, $x\odot y=b\cdot b+xaab+aaby+xaby$, and it is isomorphic to the reduced ring given in (ii).

4.

We shall give now a construction of a ring on a covering set of R.

Let $(R,[],()_\cdot)$ be an (m,n) ring and $M=\bigcup_{k\in\mathbb{N}^*} M_{k,n-1}(R)$ the set of all matrices with n-1 columns having elements from R. Define on the set M the binary relation "$\sim$" as follows:

$(a_{ij})_{\substack{i=\overline{1,r}\\ j=\overline{1,n-1}}}\sim(b_{ij})_{\substack{i=\overline{1,s}\\ j=\overline{1,n-1}}}$ if and only if $r\equiv s(\mathrm{mod}\, m-1)$ and for

$\forall c\in R\ \exists d_{ij}\in R,\ i=\overline{1,k}, j=\overline{1,n-1}$, where

$k+r\equiv 1(\mathrm{mod}\ m-1), k\in\{1,2,\dots,m-1\}$, such that

$$[(c,d_{11}^{1,n-1})_\cdot,\dots,(c,d_{k1}^{k,n-1})_\cdot,(c,a_{11}^{1,n-1})_\cdot,\dots,(c,a_{r1}^{r,n-1})_\cdot]=$$

$$=[(c,d_{11}^{1,n-1})_\cdot,\dots,(c,d_{k1}^{k,n-1})_\cdot,(c,b_{11}^{1,n-1})_\cdot,\dots,(c,b_{s1}^{s,n-1})_\cdot] \qquad (5)$$

In the expression, if needed, we understand by [...] the long sum of a number of terms congruent with 1 modulo m-1.

It is easily verified that "$\sim$" is an equivalence relation; let $R'=M_{/\sim}$ be the factor set and denote its elements by $\sum_{j=1}^{r} a_{i1}^{i,n-1}$ (this is the equivalence class of the matrix $(a_{ij})_{\substack{i=\overline{1,r}\\ j=\overline{1,n-1}}}$).

Remarks 4.1.(i)The equivalence relation may also be defined as follows:

$(a_{ij})_{\substack{i=\overline{1,r}\\ j=\overline{1,n-1}}}\sim(b_{ij})_{\substack{i=\overline{1,s}\\ j=\overline{1,n-1}}}$ if and only if $r\equiv s(\mathrm{mod}\, m-1)$ and for

$\forall c\in R\ \forall d_{ij}\in R,\ i=\overline{1,k}, j=\overline{1,n-1}$, where

$k+r\equiv 1(\mathrm{mod}\, m-1)$, $k\in\{1,2,\ldots,m-1\}$, the relation (5) holds.

(ii) An equivalent definition for the relation "$\sim$" is the following:

$(a_{ij})_{\substack{i=\overline{1,r}\\ j=\overline{1,n-1}}} \sim (b_{ij})_{\substack{i=\overline{1,s}\\ j=\overline{1,n-1}}} \Leftrightarrow r\equiv s(\mathrm{mod}\, n-1)$ and for $\forall c\in R\ \forall d_j\in R,\ j=\overline{1,n-1}$

such that $\left[\left(c^{(k-1)}, d_1^{n-1}\right)_\cdot, (\overline{c}, d_1^{n-1})_\cdot, (c, a_{11}^{1,n-1})_\cdot, \ldots, (c, a_{r1}^{r,n-1})_\cdot\right] =$

$$=\left[\left(c^{(k-1)}, d_1^{n-1}\right)_\cdot, (\overline{c}, d_1^{n-1})_\cdot, (c, b_{11}^{1,n-1})_\cdot, \ldots, (c, b_{s1}^{s,n-1})_\cdot\right] \qquad (6)$$

where again $k+r\equiv 1$ (mod $m-1$).

The proof follows immediately from the properties of the m-ary group operation.

(iii) Any permutation of the rows leads to equivalent matrices.

(iv) The equivalence class of a matrix having only one row will be denoted by $\langle a_1^{n-1}\rangle$ and is the set

$\{b_1^{n-1} \mid \forall c\in R,\ \exists d_j\in R,\ j=\overline{1,n-1}:$

$[(c^{(m-2)}, d_1^{n-1})_\cdot, (\overline{c}, d_1^{n-1})_\cdot, (c, a_1^{n-1})_\cdot] =$

$=[(c^{(m-2)}, d_1^{n-1})_\cdot, (\overline{c}, d_1^{n-1})_\cdot, (c, b_1^{n-1})_\cdot]\}$ i.e.

$$\{b_1^{n-1} \mid \forall c\in R: (c, a_1^{n-1})_\cdot = (c, b_1^{n-1})_\cdot\}. \qquad (7)$$

(v) If the (m,n) ring $(R, [\,], (\,)_\cdot)$ has a right unit u_1^{n-1} as a system of n-1 elements, then $(a_{ij})_{\substack{i=\overline{1,r}\\ j=\overline{1,n-1}}} \sim (b_{ij})_{\substack{i=\overline{1,r}\\ j=\overline{1,n-1}}}$, where $b_{ij}=u_j$, for $i=\overline{1,r},\ j=\overline{1,n-2}$ and $b_{i,n-1}=(u_{n-1}, a_{i1}^{i,n-1})_\cdot$, $i=\overline{1,r}$

(vi) Moreover, for every equivalence class $\sum_{i=1}^{r} a_{i1}^{i,m-1}$, we can choose canonical representatives having t rows, where $1 < t \leq m-1$ and $t \equiv r \pmod{m-1}$, because

$$\begin{pmatrix} a_{11} \ldots a_{1,m-1} \\ \ldots\ldots\ldots \\ a_{r1} \ldots a_{r,m-1} \end{pmatrix} \sim \begin{pmatrix} u_1 \ldots u_{m-2} & (u_{m-1}, a_{11}^{1,m-1}), \\ \ldots\ldots\ldots\ldots\ldots & \\ u_1 \ldots u_{m-2} & (u_{m-1}, a_{t-1,1}^{t-1,m-1}), \\ u_1 \ldots u_{m-2} & [(u_{m-1}, a_{t1}^{t,m-1}), \ldots, (u_{m-1}, a_{r1}^{r,m-1})] \end{pmatrix}.$$

Therefore $M/\sim = \bigcup_{k=1}^{m-1} M_{k,m-1}/\sim$.

(vii) If u_1^{m-1} is a right unit in the (m,n) ring $(R,[\,],(\,))$ then the equivalence relation "$\sim$" may also be defined as:

$(a_{ij})_{\substack{i=\overline{1,r} \\ j=\overline{1,m-1}}} \sim (b_{ij})_{\substack{i=\overline{1,s} \\ j=\overline{1,m-1}}} \Leftrightarrow r \equiv s \pmod{m-1}$ and

$[u_{m-1}^{(k-1)}, \overline{u}_{m-1}, (u_{m-1}, a_{11}^{1,m-1}), \ldots, (u_{m-1}, a_{r1}^{r,m-1})] =$

$= [u_{m-1}^{(k-1)}, \overline{u}_{m-1}, (u_{m-1}, b_{11}^{1,m-1}), \ldots, (u_{m-1}, b_{s1}^{s,m-1})]$, where $k+r \equiv 1 \pmod{m-1}$ (8)

(viii) If u_1^{n-1} is a right unit in the (m,n) ring $(R,[\,],(\,))$ we also have:

$(a_{ij})_{\substack{i=\overline{1,r} \\ j=\overline{1,m-1}}} \sim (b_{ij})_{\substack{i=\overline{1,s} \\ j=\overline{1,m-1}}} \Leftrightarrow r \equiv s \pmod{m-1}$ and for $\forall c \in R$.

$[\overset{(k-1)}{c}, \overline{c}, (c, a_{11}^{1,n-1}), \ldots, (c, a_{r1}^{r,m-1})] =$

$= [\overset{(k-1)}{c}, \overline{c}, (c, b_{11}^{1,m-1}), \ldots, (c, b_{s1}^{s,n-1})]$, where $k+r \equiv 1 \pmod{m-1}$ (9)

Example 4.2. The following matrices with m-1 rows are equivalent.

$$\begin{pmatrix} a\ldots a\ \bar{a} \\ a\ldots a\ a \\ \ldots\ldots \\ a\ldots a\ a \end{pmatrix} \sim \begin{pmatrix} b\ldots b\ \bar{b} \\ b\ldots b\ b \\ \ldots\ldots \\ b\ldots b\ b \end{pmatrix} \sim \begin{pmatrix} b\ldots\ldots b\ b \\ b\ldots\bar{b}\ldots b\ b \\ b\ldots\ldots b\ b \end{pmatrix}$$,with $\bar{b}$ an any position in the matrix.

The above matrices are also equivalent to any $(m-1)$ rows matrix of the form

(10) $$\begin{pmatrix} a_{11}\ldots a_{1m-1} \\ a_{11}\ldots a_{1m-1} \\ \ldots\ldots \\ a_{11}\ldots \bar{a}_{1m-1} \end{pmatrix},$$ where again the querelement can be placed on any position in the matrix.

We shall define now two binary operations on R^*, denoted by + and $\cdot$.
The operation $+: R^* x R^* \to R^*$ is a simple concatenation i.e.

$$\sum_{i=1}^{r} a_{i1}^{i,n-1} + \sum_{i=1}^{s} b_{i1}^{i,n-1} = \sum_{i=1}^{r+s} c_{i1}^{i,n-1},$$ where

$$c_{ij} = \begin{cases} a_{ij}, & \text{for } i=\overline{1,r} \\ b_{i-r,j}, & \text{for } i=\overline{r+1,r+s} \end{cases}, \quad j=\overline{1,n-1}$$

The operation is well defined, i.e. it does not depend on the choice of representatives. Indeed, if

$$(\alpha_{ij})_{\substack{i=\overline{1,p} \\ j=\overline{1,n-1}}} \sim (a_{ij})_{\substack{i=\overline{1,r} \\ j=\overline{1,n-1}}}, \quad (\beta_{ij})_{\substack{i=\overline{1,t} \\ j=\overline{1,n-1}}} \sim (b_{ij})_{\substack{i=\overline{1,s} \\ j=\overline{1,n-1}}}$$ and

$$\sum_{i=1}^{p} \alpha_{i1}^{i,n-1} + \sum_{i=1}^{t} \beta_{i1}^{i,n-1} = \sum_{i=1}^{p+t} \gamma_{i1}^{i,n-1},$$ then by commutativity of the m-ary operation and knowing that $p \equiv r, s \equiv t, k+r \equiv 1, l+s \equiv 1 \pmod{m-1}$ we have:

$$[(\overset{(k+l-2)}{c}, d_1^{n-1})_*, (\bar{c}, d_1^{n-1})_*, (c, \gamma_{11}^{1,n-1})_*, \ldots, (c, \gamma_{p+t,1}^{p+t,n-1})_*] =$$

$$= [\,[(\overset{(k-1)}{c}, d_1^{n-1})_*, (\bar{c}, d_1^{n-1})_*, (c, \alpha_{11}^{1,n-1})_*, \ldots, (c, \alpha_{p1}^{p,n-1})_*],$$

$$(c^{(l-1)}, d_1^{n-1})_*, (c, \beta_{11}^{1,n-1})_*, \ldots, (c, \beta_{t1}^{t,n-1})_*]^{\alpha - a} =$$

$$= [\,[(c^{(k-1)}, d_1^{n-1})_*, (\overline{c}, d_1^{n-1})_*, (c, a_{11}^{1,n-1})_*, \ldots, (c, a_{r1}^{r,n-1})_*],$$

$$(c^{(l-1)}, d_1^{n-1})_*, (c, \beta_{11}^{1,n-1})_*, \ldots, (c, \beta_{t1}^{t,n-1})_*] =$$

$$= [(c^{(k-1)}, d_1^{n-1})_*, (c, a_{11}^{1,n-1})_*, \ldots, (c, a_{r1}^{r,n-1})_*,$$

$$[(c^{(l-1)}, d_1^{n-1}), (\overline{c}, d_1^{n-1})_*, (c, \beta_{11}^{1,n-1})_*, \ldots, (c, \beta_{t1}^{t,n-1})_*]\,]^{\beta - b} =$$

$$= [\,(c^{(k-1)}, d_1^{n-1})_*, (c, a_{11}^{1,n-1})_*, \ldots, (c, a_{r1}^{1,n-1})_*,$$

$$[(c^{(l-1)}, d_1^{n-1}), (\overline{c}, d_1^{n-1})_*, (c, b_{11}^{1,n-1})_*, \ldots, (c, b_{s1}^{s,n-1})_*]\,] =$$

$$= [(c^{(k+l-2)}, d_1^{n-1})_*, (\overline{c}, d_1^{n-1})_*, (c, a_{11}^{1,n-1})_*, \ldots, (c, a_{r1}^{r,n-1})_*, (c, b_{11}^{1,n-1})_*, \ldots,$$

$$\ldots, (c, b_{s1}^{s,n-1})_*] =$$

$$= [(c^{(k+l-2)}, d_1^{n-1})_*, (\overline{c}, d_1^{n-1})_*, (c, c_{11}^{1,n-1})_*, \ldots, (c, c_{r+s,1}^{r+s,n-1})_*]$$

It is easily verified that $(R^*,+)$is an abelian group.The neuter element of this group is the class presented in example 4.2.which can be denoted by $(m-2)\langle a^{(n-1)}\rangle + \langle a^{(n-2)}, \overline{a}\rangle$.

The symmetric element of $\sum_{l=1}^{r} a_{l1}^{l,n-1}$ is

$$-\sum_{l=1}^{r} a_{l1}^{l,n-1} = (m-3)\langle a_{r1}^{r,n-1}\rangle + \langle a_{r1}^{r,n-2}, \overline{a}_{r,n-1}\rangle + \ldots + (m-3)\langle a_{11}^{1,n-1}\rangle + \langle a_{11}^{1,n-2}, \overline{a}_{1,n-1}\rangle.$$

i.e. a representative for the class $-\sum_{l=1}^{r} a_{l1}^{l,n-1}$ is the matrix:

$$\begin{pmatrix} a_{r1}\ldots a_{r,n-1} \\ \ldots\ldots\ldots \\ a_{r1}\ldots a_{r,n-1} \\ a_{r1}\ldots \overline{a}_{r,n-1} \\ \ldots\ldots\ldots \\ a_{11}\ldots a_{1,n-1} \\ \ldots\ldots\ldots \\ a_{11}\ldots a_{1,n-1} \\ a_{11}\ldots \overline{a}_{1,n-1} \end{pmatrix}$$

The second binary operation $\cdot : R^{*}\times R^{*}\to R^{*}$ will be defined as follows:

$$\sum_{i=1}^{r} a_{i1}^{i,n-1}\cdot\sum_{l=1}^{s} b_{l1}^{l,n-1}=\sum_{j=1}^{s}\sum_{i=1}^{r} a_{i1}^{i,n-2}\,(a_{i,n-1},b_{j1}^{j,n-1})_{\cdot}$$

The definition does not depend on the choice of representatives; indeed,making the same notations as before and putting v+rs=1(mod m-1) we have:

$$[(\overset{(v-1)}{c},d_1^{n-1})_{+},(\overline{c},d_1^{n-1})_{+},(c,a_{11}^{1,n-2},(a_{1,n-1},b_{11}^{1,n-1})_{\cdot})_{\cdot},\ldots,$$

$$(c,a_{11}^{1,n-2},(a_{1,n-1},b_{s1}^{s,n-1})_{\cdot})_{\cdot},\ldots,(c,a_{r1}^{r,n-2},(a_{r,n-1},b_{11}^{1,n-1})_{\cdot})_{\cdot},\ldots,$$

$\ldots,(c,a_{r1}^{r,n-2},(a_{r,n-1},b_{s1}^{s,n-1})_{\cdot})_{\cdot}]=$ by the distributive laws, querelement's properties and commutativity=

$$=[(\overset{(v-1)}{c},d_1^{n-1})_{+},(\overline{c},d_1^{n-1})_{+},(c,a_{11}^{1,n-2},[(\overset{(l-1)}{a_{1,n-1}},d_1^{n-1})_{+},$$

$$(\overline{a}_{1,n-1},d_1^{n-1})_{+},(a_{1,n-1},b_{11}^{1,n-1})_{\cdot},\ldots,(a_{1,n-1},b_{s1}^{s,n-1})_{\cdot}])_{\cdot},$$

$$(c,\overset{(m-l-1)}{a_{11}^{1,n-1}},d_1^{n-1})_{\cdot},\ldots,(c,a_{r1}^{r,n-2},[(\overset{(l-1)}{a_{r,n-1}},d_1^{n-1})_{+},$$

$$(\overline{a}_{r,n-1},d_1^{n-1})_{+},(a_{r,n-1},b_{11}^{1,n-1})_{\cdot},\ldots,(a_{r,n-1},b_{s}^{s,n-1})_{\cdot}]),$$

$$(c,\overset{(m-l-1)}{a_{r1}^{r,n-1}},d_1^{n-1})_{\cdot}]\overset{\beta-b}{=}[(\overset{(v-1)}{c},d_1^{n-1})_{+},(\overline{c},d_1^{n-1})_{+},$$

$(c, a_{11}^{1,n-2}, [(a_{1,n-1} \overset{(l-1)}{,} d_1^{n-1})_*,$

$(\overline{a}_{1,n-1}, d_1^{n-1})_*, (a_{1,n-1}, \beta_{11}^{1,n-1})_*, \ldots, (a_{1,n-1}, \beta_{tl}^{t,n-1})_*])_*,$

$(c, a_{11}^{1,n-1} \overset{(m-l-1)}{,} d_1^{n-1})_*, \ldots, (c, a_{rl}^{r,n-2}, [(a_{r,n-1} \overset{(l-1)}{,} d_1^{n-1})_*, (\overline{a}_{r,n-1}, d_1^{n-1})_*,$

$(a_{r,n-1}, \beta_{11}^{1,n-1})_*, \ldots, (a_{r,n-1}, \beta_{tl}^{t,n-1})_*], (c, a_{rl}^{r,n-1} \overset{(m-l-1)}{,} d_1^{n-1})_*] =$

again by the distributivity laws, querelement's properties and commutativity=

$=[(c \overset{(v-1)}{,} d_1^{n-1})_*, (\overline{c}, d_1^{n-1})_*, (c, a_{11}^{1,n-2}(a_{1,n-1}, \beta_{11}^{1,n-1})_*)_*, \ldots,$

$\ldots, (c, a_{11}^{1,n-2}(a_{1,n-1}, \beta_{tl}^{t,n-1})_*)_*, \ldots, (c, a_{rl}^{r,n-2}(a_{r,n-1}, \beta_{11}^{1,n-1})_*)_*, \ldots,$

$\ldots, (c, a_{rl}^{r,n-2}(a_{r,n-1}, \beta_{tl}^{t,n-1})_*)_*] =$ by associative and distributive laws=

$=[(c \overset{(v-1)}{,} d_1^{n-1})_*, (\overline{c}, d_1^{n-1})_*, ([(c \overset{(k-1)}{,} d_1^{n-1})_*, (\overline{c}, d_1^{n-1})_*, (c, a_{11}^{1,n-1})_*, \ldots,$

$(c, a_{rl}^{r,n-1})_*], \beta_{11}^{1,n-1})_*, (c, d_1^{n-1} \overset{(m-k-1)}{,} \beta_{11}^{1,n-1})_*, \ldots, (c, d_1^{n-1} \overset{(m-k-1)}{,} \beta_{tl}^{t,n-1})_*,$

$([(c \overset{(k-1)}{,} d_1^{n-1})_*, (\overline{c}, d_1^{n-1})_*, (c, a_{11}^{1,n-1})_*, \ldots,$

$\ldots, (c, a_{rl}^{r,n-1})_*], \beta_{tl}^{t,n-1})_*] \overset{*-*}{=}$

$=[(c \overset{(v-1)}{,} d_1^{n-1})_*, (\overline{c}, d_1^{n-1}), ([(c \overset{(k-1)}{,} d_1^{n-1})_*, (\overline{c}, d_1^{n-1})_*, (c, \alpha_{11}^{1,n-1})_*, \ldots,$

$\ldots, (c, \alpha_{pl}^{p,n-1})_*], \beta_{11}^{1,n-1})_*, (c, d_1^{n-1} \overset{(m-k-1)}{,} \beta_{11}^{1,n-1})_*, \ldots, (c, d_1^{n-1} \overset{(m-k-1)}{,} \beta_{tl}^{t,n-1})_*,$

$([(c \overset{(k-1)}{,} d_1^{n-1})_*, (\overline{c}, d_1^{n-1})_*, (c, \alpha_{11}^{1,n-1})_*, \ldots, (c, \alpha_{pl}^{p,n-1})_*], \beta_{tl}^{t,n-1})_*] =$ by the

distributive laws, querelement's properties and commutativity of the m-ary operation

$=[(c^{(v-1)}, d_1^{n-1})_*, (\overline{c}, d_1^{n-1})_*, (c, \alpha_{11}^{1,n-2}, (\alpha_{1,n-1}, \beta_{11}^{1,n-1})_*)_*, \ldots,$

$\ldots, (c, \alpha_{11}^{1,n-2}, (\alpha_{1,n-1}, \beta_{t1}^{t,n-1})_*)_*, \ldots, (c, \alpha_{p1}^{p,n-2}, (\alpha_{p,n-1}, \beta_{11}^{1,n-2})_*)_*, \ldots,$

$\ldots, (c, \alpha_{p1}^{p,n-2}, (\alpha_{p,n-1}, \beta_{t1}^{t,n-1})_*)_*]$ which proves that

$$\sum_{i=1}^{r} a_{i1}^{i,n-1} \cdot \sum_{i=1}^{s} b_{i1}^{i,n-1} = \sum_{i=1}^{p} \alpha_{i1}^{i,n-1} \cdot \sum_{i=1}^{t} \beta_{i1}^{i,n-1}.$$

The associative law for the operation "$\cdot$" is an immediate consequence of the associative laws for the n-ary operation $()_\circ$. If u_1^{n-1} is a right unit in the (m,n) ring $(R,[],()_*)$ then its equivalence class $\langle u_1^{n-1} \rangle$, consisting of all the right units of the (m,n) ring $(R,[],()_*)$, is a right unit for the operation "$\cdot$".

Indeed, $\sum_{i=1}^{r} a_{i1}^{i,n-1} \cdot \langle u_1^{n-1} \rangle = \sum_{i=1}^{r} a_{i1}^{i,n-2} (a_{i,n-1} u_1^{n-1})_* = \sum_{i=1}^{r} a_{i1}^{i,n-1}$.

If the (m,n) ring $(R,[],()_*)$ is semicommutative, then for any $2n-1$ elements of R we have:

$$((c, a_1^{n-1})_*, b_1^{n-1})_* = ((c, b_1^{n-1})_*, a_1^{n-1})_*.$$

This equality implies the commutativity of multiplication in R^*, so we conclude that if $(R,[],()_*)$ is a semicommutative (m,n) ring then $(R^*,\cdot)$ is a commutative semigroup.

In R^* the distributive laws also hold; indeed:

$$\sum_{i=1}^{r} a_{i1}^{i,n-1} \cdot \left(\sum_{i=1}^{s} b_{i1}^{i,n-1} + \sum_{i=1}^{t} c_{i1}^{i,n-1} \right) = \sum_{i=1}^{r} a_{i1}^{i,n-1} \cdot \sum_{i=1}^{s+t} d_{i1}^{i,n-1} =$$

$$= \sum_{j=1}^{s+t} \sum_{i=1}^{r} a_{i1}^{i,n-2} (a_{i,n-1}, d_{j1}^{j,n-1})_* =$$

$$=\sum_{j=1}^{s}\sum_{i=1}^{r} a_{i1}^{i,n-2}\left(a_{i,n-1}, d_{j1}^{j,n-1}\right)_{*}+\sum_{j=s+1}^{s+t}\sum_{i=1}^{r} a_{i1}^{i,n-2}\left(a_{i,n-1}, d_{j1}^{j,n-1}\right)_{*}=$$

$$=\sum_{j=1}^{s}\sum_{i=1}^{r} a_{i1}^{i,n-2}\left(a_{i,n-1}, b_{j,1}^{j,n-1}\right)_{*}+\sum_{j=1}^{t}\sum_{i=1}^{r} a_{i1}^{i,n-2}\left(a_{i,n-1}, c_{j1}^{j,n-1}\right)_{*}=$$

$$=\sum_{i=1}^{r} a_{i1}^{i,n-1}\cdot\sum_{i=1}^{s} b_{i1}^{i,n-1}+\sum_{i=1}^{r} a_{i1}^{i,n-1}\cdot\sum_{i=1}^{t} c_{i1}^{i,n-1}$$

(since $d_{ij}=b_{ij}$, for $i=\overline{1,s}, j=\overline{1,n-1}$ and $d_{ij}=c_{i-s,j}$, for $i=\overline{s+1,s+t}, j=\overline{1,n-1}$)

We have proved so the following

Proposition 4.3. $(R^{*},+,\cdot)$ *is a ring. If* $(R,[\,],(\,)_{*})$ *is a* (m,n) *ring with right unit then* $(R^{*},+,\cdot)$ *has a right unit;if* $(R,[\,],(\,)_{*})$ *is a semicommutative* (m,n) *ring then* $(R^{*},+,\cdot)$ *is a commutative ring.*

Proposition 4.4. *The equivalence classes having matrices of type* $(r,n-1)$, *with* $r\equiv 0\ (\mathrm{mod}\ m-1)$, *as representatives form an ideal* I *of* $(R^{*},+,\cdot)$, *and* $R^{*}/I\cong \mathbb{Z}_{m-1}$.

Proof Let $\sum_{i=1}^{r} a_{i1}^{i,n-1}, \sum_{i=1}^{s} b_{i1}^{i,n-1}\in I$, i.e. $r\equiv s\equiv 0\ (\mathrm{mod}\ m-1)$.

Their sum has a representative consisting of $r+s\equiv 0\ (\mathrm{mod}\ m-1)$ ordered systems of $(n-1)$ elements i.e. it belongs to I.

We have that $-\sum_{i=1}^{s} b_{i1}^{i,n-1}=(m-3)\langle b_{s1}^{s,n-1}\rangle+\langle b_{s,1}^{s,n-2}, \bar{b}_{s,n-1}\rangle+\ldots+$

$+\ldots+(m-3)\langle b_{11}^{1,n-1}\rangle+\langle b_{11}^{1,n-2}, \bar{b}_{1,n-1}\rangle$;

this representative has $s(m-2)$ rows -a number congruent to 0 modulo $m-1$. Hence $-\sum_{i=1}^{s} b_{i1}^{i,n-1}\in I$.

Finally for any $\sum_{i=1}^{r} a_{i1}^{i,n-1} \in R$ and $\sum_{i=1}^{s} b_{i1}^{i,n-1} \in I$ we have that their product

$\sum_{i=1}^{r} a_{i1}^{i,n-1} \cdot \sum_{i=1}^{s} b_{i1}^{i,n-1}$ has a representative with $rs \equiv 0 \pmod{m-1}$ rows, so it belongs to I.

Proposition 4.5 *If* $a u_1^{n-2}$ *is a left unit and* $u_1^{n-2} a$ *is a right unit in the* (m,n) *ring* $(R,[],()_\cdot)$ *then the ideal* I *is isomorphic to the reduced ring* $red_a'(red_{u_1^{n-2}}(R,[],()_\cdot))$.

Proof Let $f: I \to R$, $f(\sum_{i=1}^{r} a_{i1}^{i,n-1}) = [a, (a, a_{11}^{1,n-1})_\cdot, \ldots, (a, a_{r1}^{r,n-1})_\cdot]$.

$$f(\sum_{i=1}^{r} a_{i1}^{i,n-1} + \sum_{i=1}^{s} b_{i1}^{i,n-1}) =$$

$$= [a, (a, a_{11}^{1,n-1})_\cdot, \ldots, (a, a_{r1}^{r,n-1})_\cdot, (a, b_{11}^{1,n-1})_\cdot, \ldots, (a, b_{s1}^{s,n-1})_\cdot] =$$

$$= [a, (a, a_{11}^{1,n-1})_\cdot, \ldots, (a, a_{r1}^{r,n-1})_\cdot, \overset{(m-3)}{a}, \overline{a}, a, (a, b_{11}^{1,n-1})_\cdot, \ldots, (a, b_{s1}^{s,n-1})_\cdot] =$$

$$= [a, (a, a_{11}^{1,n-1})_\cdot, \ldots, (a, a_{r1}^{r,n-1})_\cdot] \oplus$$

$$\oplus [a, (a, b_{11}^{1,n-1})_\cdot, \ldots, (a, b_{s1}^{s,n-1})_\cdot] = f\left(\sum_{i=1}^{r} a_{i1}^{i,n-1}\right) \oplus f\left(\sum_{i=1}^{s} b_{i1}^{1,n-1}\right)$$

$$f\left(\sum_{i=1}^{r} a_{i1}^{i,n-1} \cdot \sum_{i=1}^{s} b_{i1}^{i,n-1}\right) = [a, (a, a_{11}^{1,n-1}, b_{11}^{1,n-1})_\cdot, \ldots,$$

$$\ldots, (a, a_{11}^{1,n-1}, b_{s1}^{s,n-1})_\cdot, \ldots, (a, a_{r1}^{r,n-1}, b_{11}^{1,n-1})_\cdot, \ldots, (a, a_{r1}^{r,n-1}, b_{s1}^{s,n-1})_\cdot]$$

$$f\left(\sum_{i=1}^{r} a_{i1}^{i,n-1}\right) \odot f\left(\sum_{i=1}^{s} b_{i1}^{i,n-1}\right) = [a, a, \overset{(m-3)}{a}, (\overset{(n-1)}{a}, a_{11}^{1,n-1})_\cdot, \ldots,$$

$$(\overset{(n-1)}{a}, a_{r1}^{r,n-1})_\cdot, \overline{a}, (\overline{a}, a_{11}^{1,n-1})_\cdot, \ldots, (\overline{a}, a_{r1}^{r,n-1})_\cdot, \overset{(m-3)}{a}, (\overset{(n-1)}{a}, b_{11}^{1,n-1})_\cdot, \ldots,$$

$$\ldots,(\overset{(m-3)}{a},b_{s1}^{s,n-1})_{\circ},\overline{a},(\overline{a},b_{11}^{1,n-1})_{\circ},\ldots,(\overline{a},b_{s1}^{s,n-1})_{\circ},([a,(a,a_{11}^{1,n-1})_{\circ},\ldots$$

$$\ldots,(a,a_{11}^{r,n-1})_{\circ}],u_1^{n-2},[a,(a,b_{11}^{1,n-1})_{\circ},\ldots,(a,b_{s1}^{s,n-1})_{\circ}])_{\circ}]=$$

$$=[(\overset{(m-3)}{a},a_{11}^{1,n-1})_{\circ},\ldots,(\overset{(m-3)}{a},a_{r1}^{r,n-1})_{\circ},(\overline{a},a_{11}^{1,n-1})_{\circ},\ldots,(\overline{a},a_{11}^{r,n-1})_{\circ},(\overset{(m-3)}{a},b_{11}^{1,n-1})_{\circ},$$

$$\ldots,(\overset{(m-3)}{a},b_{s1}^{s,n-1})_{\circ},(\overline{a},b_{11}^{1,n-1})_{\circ},\ldots,(\overline{a},b_{s1}^{s,n-1})_{\circ},a,(a,a_{11}^{1,n-1})_{\circ},\ldots$$

$$(a,a_{r1}^{r,n-1})_{\circ},(a,b_{11}^{1,n-1})_{\circ},\ldots,(a,b_{s1}^{s,n-1})_{\circ},(a,a_{11}^{1,n-1},b_{11}^{1,n-1})_{\circ},\ldots,$$

$$\ldots,(a,a_{11}^{1,n-1},b_{11}^{s,n-1})_{\circ},\ldots,(a,a_{r1}^{r,n-1})_{\circ},\ldots,(a,a_{r1}^{r,n-1},b_{s1}^{s,n-1})_{\circ}]=$$

$$=[a,(a,a_{11}^{1,n-1},b_{11}^{1,n-1})_{\circ},\ldots,(a,a_{11}^{1,n-1},b_{s1}^{s,n-1})_{\circ},\ldots,(a,a_{r1}^{r,n-1},b_{11}^{1,n-1})_{\circ},\ldots$$

$$\ldots,(a,a_{r1}^{r,n-1},b_{s1}^{s,n-1}]=f\left(\sum_{i=1}^{r}a_{i1}^{i,n-1}\cdot\sum_{i=1}^{s}b_{i1}^{i,n-1}\right).$$

We have proved that f is a homomorphism of rings.

For $\forall\, x\in R$ we have that

$$x=[a,(a,u_1^{n-2},a)_{\circ},\ldots,(a,u_1^{n-2},\overline{a})_{\circ},(a,u_1^{n-2},x)_{\circ}]=f\left(\sum_{i=1}^{m-1}b_{i1}^{i,n-1}\right)\text{, where}$$

$b_{ij}=u_j$ for $j=\overline{1,n-2}$, $i=\overline{1,m-1}$, $b_{i,n-1}=a$ for $i=\overline{1,m-3}$, $b_{m-2,n-1}=\overline{a}$, $b_{m-1,n-1}=x$.

This proves that f is onto;the fact that f is one-to-one is obvious.

Remark. Note that the set $\langle u_1^{n-2}R\rangle$ together with the repeated operation $\oplus$ and with $\odot$ is isomorphic to the $(m,2)$ ring $red_{u_1^{n-2}}(R,[\,],(\,)_{\circ})$ defined in [6].

Example 4.6. (i) Let $(\mathbb{Z}_6,[\,],(\,)_{\circ})$ the (3,3) commutative ring given in example 3.3.(i).

Then the element of the ring $(\mathbb{Z}_6^*,+,\cdot)$ will be the following equivalence classes:

$<\hat{3},\hat{0}>=\{(\hat{2},\hat{4}),(\hat{4},\hat{4}),(\hat{6},\hat{4}),(\hat{a},\hat{0}),\hat{a}\in\mathbb{Z}_8\}$,

$<\hat{3},\hat{3}>=\{(\hat{3},\hat{3}),(\hat{5},\hat{5}),(\hat{7},\hat{7}),(\hat{1},\hat{1})\}$,

$<\hat{3},\hat{6}>=\{(\hat{3},\hat{6}),(\hat{1},\hat{2}),(\hat{2},\hat{5}),(\hat{6},\hat{7})\}$,

$<\hat{3},\hat{1}>=\{(\hat{3},\hat{1}),(\hat{5},\hat{7})\}$

$<\hat{3},\hat{4}>=\{(\hat{3},\hat{4}),(\hat{1},\hat{4}),(\hat{2},\hat{2}),(\hat{2},\hat{6}),(\hat{4},\hat{5}),(\hat{6},\hat{6}),(\hat{4},\hat{7})\}$,

$<\hat{3},\hat{7}>=\{(\hat{3},\hat{7}),(\hat{1},\hat{5})\}$

$<\hat{3},\hat{2}>=\{(\hat{3},\hat{2}),(\hat{1},\hat{6}),(\hat{2},\hat{7}),(\hat{5},\hat{6})\}$

$<\hat{3},\hat{5}>=\{(\hat{3},\hat{5}),(\hat{1},\hat{7})\}$.

$$\sum\begin{pmatrix}\hat{3} & \hat{0}\\ \hat{3} & \hat{0}\end{pmatrix},\ \sum\begin{pmatrix}\hat{3} & \hat{0}\\ \hat{3} & \hat{3}\end{pmatrix},\ \sum\begin{pmatrix}\hat{3} & \hat{0}\\ \hat{3} & \hat{6}\end{pmatrix},\ \sum\begin{pmatrix}\hat{3} & \hat{0}\\ \hat{3} & \hat{1}\end{pmatrix},$$

$$\sum\begin{pmatrix}\hat{3} & \hat{0}\\ \hat{3} & \hat{4}\end{pmatrix},\ \sum\begin{pmatrix}\hat{3} & \hat{0}\\ \hat{3} & \hat{7}\end{pmatrix},\ \sum\begin{pmatrix}\hat{3} & \hat{0}\\ \hat{3} & \hat{2}\end{pmatrix},\ \sum\begin{pmatrix}\hat{3} & \hat{0}\\ \hat{3} & \hat{5}\end{pmatrix},$$

where $\sum\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}=\left\{\begin{pmatrix}y_1 & y_2\\ y_3 & y_4\end{pmatrix}\middle| x_1x_2+x_3x_4=y_1y_2+y_3y_4\right\}$

The isomorphism defined in proposition 4.5 is: $f:I\to\mathbb{Z}_8^*$

$$f\left(\sum\begin{pmatrix}\hat{3} & \hat{0}\\ \hat{3} & \hat{0}\end{pmatrix}\right)=\hat{3},\ f\left(\sum\begin{pmatrix}\hat{3} & \hat{0}\\ \hat{3} & \hat{3}\end{pmatrix}\right)=\hat{4},$$

$$f\left(\sum\begin{pmatrix}\hat{3} & \hat{0}\\ \hat{3} & \hat{6}\end{pmatrix}\right)=\hat{5},\ f\left(\sum\begin{pmatrix}\hat{3} & \hat{0}\\ \hat{3} & \hat{1}\end{pmatrix}\right)=\hat{6},$$

$$f\left(\sum\begin{pmatrix}\hat{3} & \hat{0}\\ \hat{3} & \hat{4}\end{pmatrix}\right)=\hat{7},\ f\left(\sum\begin{pmatrix}\hat{3} & \hat{0}\\ \hat{3} & \hat{7}\end{pmatrix}\right)=\hat{0},$$

$$f\left(\sum\begin{pmatrix}\hat{3} & \hat{0}\\ \hat{3} & \hat{2}\end{pmatrix}\right)=\hat{1},\ f\left(\sum\begin{pmatrix}\hat{3} & \hat{0}\\ \hat{3} & \hat{5}\end{pmatrix}\right)=\hat{2}$$

(ii)Let R be the (3,4)commutative ring defined in example 3.3.(ii). The elements of the ring $(R',+,\cdot)$ will be $<abb>=\{(abb),(aaa)\}$,

$\langle aba\rangle=\{(aba),(bbb)\}$.

$$\sum\begin{pmatrix} a & b & a \\ a & b & a \end{pmatrix}=\left\{\begin{pmatrix} a & b & a \\ a & b & a \end{pmatrix},\begin{pmatrix} b & b & b \\ b & b & b \end{pmatrix},\begin{pmatrix} a & b & a \\ b & b & b \end{pmatrix},\begin{pmatrix} a & a & a \\ a & a & a \end{pmatrix},\begin{pmatrix} a & b & b \\ a & b & b \end{pmatrix},\begin{pmatrix} a & a & a \\ a & b & b \end{pmatrix}\right\}$$

$$\sum\begin{pmatrix} a & b & a \\ a & b & b \end{pmatrix}=\left\{\begin{pmatrix} a & b & a \\ a & b & b \end{pmatrix},\begin{pmatrix} a & a & b \\ a & a & a \end{pmatrix},\begin{pmatrix} a & b & b \\ b & b & b \end{pmatrix},\begin{pmatrix} a & a & a \\ b & b & b \end{pmatrix}\right\}$$

$I=\left\{\sum\begin{pmatrix} a & b & a \\ a & b & a \end{pmatrix},\ \sum\begin{pmatrix} a & b & a \\ a & b & b \end{pmatrix}\right\}$. The isomorphism will be : $f:I\to R'$,

$$f\left(\sum\begin{pmatrix} a & b & a \\ a & b & a \end{pmatrix}\right)=b,\quad f\left(\sum\begin{pmatrix} a & b & a \\ a & b & b \end{pmatrix}\right)=a$$

(iii)Let R be the (3,4) commutative ring given in example 3.3(iii). The elements of the ring $(R^*,+,\cdot)$ will be: $\langle abb\rangle=\{(abb),(aaa)\}$, $\langle aba\rangle=\{(aba),(bbb)\}$

$$\sum\begin{pmatrix} abc \\ aba \end{pmatrix}=\left\{\begin{pmatrix} a & b & a \\ a & b & a \end{pmatrix},\begin{pmatrix} a & a & a \\ a & a & a \end{pmatrix},\begin{pmatrix} a & b & b \\ a & b & b \end{pmatrix},\begin{pmatrix} a & a & a \\ a & b & b \end{pmatrix},\begin{pmatrix} a & a & b \\ b & b & b \end{pmatrix},\begin{pmatrix} b & b & b \\ b & b & b \end{pmatrix}\right\}$$

$$\sum\begin{pmatrix} aba \\ abb \end{pmatrix}=\left\{\begin{pmatrix} a & b & a \\ a & b & b \end{pmatrix},\begin{pmatrix} a & a & a \\ a & b & a \end{pmatrix},\begin{pmatrix} a & a & a \\ b & b & b \end{pmatrix},\begin{pmatrix} a & b & b \\ b & b & b \end{pmatrix}\right\}$$

$I=\left\{\sum\begin{pmatrix} a & b & a \\ a & b & a \end{pmatrix},\sum\begin{pmatrix} a & b & a \\ a & b & b \end{pmatrix}\right\}$ and the isomorphism f is $f:I\to R^*$

$$f\left(\sum\begin{pmatrix} a & b & a \\ a & b & a \end{pmatrix}\right)=a,\quad f\left(\sum\begin{pmatrix} a & b & a \\ a & b & b \end{pmatrix}\right)=b$$

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University of Baia Mare
Victoriei 76, 4800 Baia Mare
ROMANIA