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ON THE APPROXIMATION OF THREEVARIATE B-CONTINUOUS FUNCTIONS

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Abstract: The goal of the paper is to extend the results of a paper by C.Badea, I.Badea and H.H.Gonska [1] to the approximation of the threevariate B-continuous functions in the sense of K.Bögel [5]. In section 1, the notions of B-continuous and uniformly B-continuous function from [5] are extend to the case of the threevariate B-continuous functions. Some relationship among these notions are established.

In section 2 we discuss a Korovkin type criterion for the approximation by means of linear positive operators of the B-continuous functions of three variables.

In section 3 we present some applications illustrated by the approximation of threevariate functions using the operators of Bernstein -Stancu, and Bleiman-Butzer-Hahn.

1. THREEVARIATE B-CONTINUOUS FUNCTIONS

Let us denote by $\mathbb{R}^{I^k}$ the set of functions $f:I^k\to\mathbb{R}$ where $I=[0,1]$ and k is a positive integer.

The notion of B-continuous function was defined by K. Bögel[5], using the operator $\Delta_2:\mathbb{R}^{I^2}\to\mathbb{R}^{I^2}$ given by:

$$(1.1)\quad \Delta_2[f;M,M'] \underset{\text{notation}}{=} \Delta_{s,t}[f;x,y]=f(s,t)-f(x,t)-f(s,y)+f(x,y)$$

for any function $f\in\mathbb{R}^{I^2}$ and any points $M(x,y), M'(s,t)\in I^2$.

Let us to denote by $\Delta:\mathbb{R}^{I}\to\mathbb{R}^{I}$ the operator defined by:

$$(1.2)\quad \Delta[f;M,M'] \underset{\text{notation}}{=} \Delta_s[f;x]=f(s)-f(x)$$

for any functions $f\in\mathbb{R}^{I}$ and any points $M(x), M'(s)\in I$.

If $f\in\mathbb{R}^{I^2}$, we denote by ${}_x\Delta, {}_y\overline{\Delta}$ the parametrical execusions of operator Δ defined at (1.2)and we observe that the equality:

$$(1.3)\quad \Delta_{s,t}[f;x,y]=({}_x\Delta\circ{}_y\overline{\Delta})[f;x,y]$$

is true.

The above remark permits us to define the operator of threevariate difference by:

Definition 1.1.:Let be ${}_x\Delta, {}_y\overline{\Delta}, {}_z\overline{\Delta}$

the parametrical extensions of the operator (1.2).The operator $\Delta_3:\mathbb{R}^{I^3}\to\mathbb{R}^{I^3}$ given by:

$$(1.4)\quad \Delta_3[f;M,M'] \underset{\text{notation}}{=} \Delta_{s,t,u}[f;x,y,z]=({}_x\Delta\circ{}_y\overline{\Delta}\circ{}_z\overline{\Delta})[f;x,y,z]$$

is called operator of threevariate difference.

Remark 1.1:It is easy to see that the representation:

(1.5) $\Delta_3[f;M,M'] = \Delta_{s,t,u}[f;x,y,z] =$

$$= f(s,t,u) - f(s,y,z) - f(x,t,z) - f(x,y,u) +$$
$$+ f(s,t,z) + f(s,y,u) + f(x,t,u) - f(x,y,z)$$

is valid.

Definition 1.2: Let be $M(x,y,z)$ a fixed point I^3. A function $f \in \mathbb{R}^{I^3}$ is called B- continuous on $M(x,y,z)$ if the equality:

(1.6) $$\lim_{(s,t,u)\to(x,y,z)} \Delta_{s,t,u}[f;x,y,z] = 0$$

holds. If $f \in \mathbb{R}^{I^3}$ is B-continuous on every point of I^3 we say that f is B - continuous on I^3. We denote be $C_b(I^3)$ the set of B-continuous functions on I^3

The relationship between B-continuous functions and usual continuous functions is contained in:

Lemma 1.1: *If $f \in C_b(I^3)$, the function $g \in \mathbb{R}^{I^3}$, defined by:*

(1.7) $$g(s,t,u) = f(s,t,u) - f(s,y,z) - f(x,t,z) - f(x,y,u) +$$
$$+ f(s,t,z) + f(s,y,u) + f(x,t,u),$$

is continuous in I^3.

Proof: Let be $(x,y,z) \in I^3$ a fixed point and $(s,t,u) \in I^3$ a variable point. From (1.7) it follows the equality:

(1.8) $g(x,y,z) = f(x,y,z)$

On the other hand, using the operator (1.5) we obtain that the function given at (1.7) can be represented under the form:

(1.9) $g(s,t,u) = \Delta_{s,t,u}[f;x,y,z] + f(x,y,z)$

Taking into account that $f \in C_b(I^3)$, from (1.9) we deduce:

(1.10)

$$\lim_{(s,t,u)\to(x,y,z)} g(s,t,u) = \lim_{(s,t,u)\to(x,y,z)} (\Delta_{s,t,u}[f;x,y,z] + f(x,y,z)) = g(x,y,z)$$

wich shows that q is B-continuous on (x,y,z). Because (x,y,z) was arbitrary chosen in I^3, it results that g is continuous in I^3.

Definition 1.2: The function $f\in\mathbb{R}^{I^3}$ is namely uniform B-continuous in I^3 if for $(\forall)\epsilon>0$ there exists $\delta=\delta(\epsilon)>0$ such that for every points $(s,t,u),(x,y,z)\in I^3$ for wich we have:

(1.11) $|s-x|<\delta,\ |y-t|<\delta,\ |z-u|<\delta$

the inequality:

(1.12) $|\Delta_{s,t,u}[f;x,y,z]|<\epsilon$

holds.

The relationship between B - cotinuous functions and uniform B-continuous functions is given in:

Lemma 1.2:*If $f\in C_b(I^3)$, then f is uniform B-continuous on I^3.*

Proof:If g is the function given at (1.7), by (1.8)and(1.9)we obtain:

(1.13) $\Delta_{s,t,u}[f;x,y,z]=g(s,t,u)-g(x,y,z)$

From the lemma 1.1, there follows that q is continuous in I^3. Because I^3 is a compact subset of $\mathbb{R}^3$, it results that g is uniform continuous on I^3. It follows that $(\forall)\epsilon>0$ there exists $\delta=\delta(\epsilon)>0$ such that for every $(x,y,z),(s,t,u)\in I^3$, satisfaying (1.11), we have:

(1.14) $|g(s,t,u)-g(x,y,z)|<\epsilon$

From (1.13) and (1.14)it results that f satisfies (1.12),i.e.f is uniform B- continuous on I^3.

Definition 1.3: The function $f\in\mathbb{R}^{I^3}$ is caled B-bounded on I^3 if there exists a positive number K such that:

(1.5) $|\Delta_{s,t,u}[f;x,y,z]| \leq K$

for every two points $(s,t,u),(x,y,z)$ of I^3

Lemma 1.3: *If $f \in C_b(I^3)$, then f is B-bounded on I^3*

Proof: The function g given at (1.7) is continuous on I^3. Because I^3 is a compact subset of $\mathbb{R}^3$, it results that f is bounded in the usual sence on I^3. Let us introduce the notation $M = \max_{(s,t,u) \in I^3} g(s,t,u)$; from (1.8) and (1.9) it result that the inequality:

(1.16) $|\Delta_{s,t,u}[f;x,y,z]| = |g(s,t,u) - g(x,y,z)| \leq 2|M|$

is true for every points $(s,t,u),(x,y,z) \in I^3$. The inequality (1.15) is also valid with $K=2|M|$.

2. A KOROVKIN TYPE THEOREM FOR APPROXIMATION IN $B(I^3)$.

In this section we shall prove an analogue of the Korovkin type theorem given in [1] for the approximation in $B(I^3)$. First, we establish an auxilliary result, corresponding to a result from [1], given by:

Lemma 2.1: *Let be $f \in C_b(I^3)$ arbitrarly chosen. For every positive number ϵ there are three positive numbers $A(\epsilon) = A(\epsilon, f)$, $B(\epsilon) = B(\epsilon, f)$, $C(\epsilon) = C(\epsilon, f)$ such that for every $(x,y,z),(s,t,u) \in I^3$ we have:*

(2.1) $|\Delta_{s,t,u}[f;x,y,z]| \leq \frac{\epsilon}{4} + A(\epsilon)(x-s)^2 + B(\epsilon)(y-t)^2 + C(\epsilon)(z-u)^2$

Proof: Because f is from $C_b(I^3)$, the lemma 1.2 permets to deduce that f is also uniform B-continous on I^3, that is for each

$(x,y,z),(s,t,u)\in I^3$ with $|x-s|<\delta(\epsilon),|(y-t)|<\delta(\epsilon),|z-u|<\delta(\epsilon)$ we have:

(2.2) $|\Delta_{s,t,u}[f;x,y,z]|<\frac{\epsilon}{4}$

Let be ϵ a given positive number and $(x,y,z),(s,t,u)\in I^3$. We shall investigate the following eight situations:

(i) $|x-s|<\delta(\epsilon),|y-t|<\delta(\epsilon),|z-u|<\delta(\epsilon)$

(ii) $|x-s|<\delta(\epsilon),|y-t|<\delta(\epsilon),|z-u|\geq\delta(\epsilon)$

(iii) $|x-s|<\delta(\epsilon),|y-t|\geq\delta(\epsilon),|z-u|<\delta(\epsilon)$

(iv) $|x-s|\geq\delta(\epsilon),|y-t|<\delta(\epsilon),|z-u|<\delta(\epsilon)$

(v) $|x-s|<\delta(\epsilon),|y-t|\geq\delta(\epsilon),|z-u|\geq\delta(\epsilon)$

(vi) $|x-s|\geq\delta(\epsilon),|y-t|<\delta(\epsilon),|z-u|\geq\delta(\epsilon)$

(vii) $|x-s|\geq\delta(\epsilon),|y-t|\geq\delta(\epsilon),|z-u|<\delta(\epsilon)$

(viii) $|x-s|\geq\delta(\epsilon),|y-t|\geq\delta(\epsilon),|z-u|\geq\delta(\epsilon)$

In the case (i), by (2.2) we can conclude that we have:

(2.3) $|\Delta_{s,t,u}[f;x,y,z]|\leq\frac{\epsilon}{4}$

Now let us consider the case (ii). Because f is B - continous on I^3, there is (see the lemma 1.3) a positive number M such that we have:

(2.4) $|\Delta_{s,t,u}[f;x,y,z]|\leq M$

From (2.4)and the third inequality of (ii) we obtain that:

(2.5) $|\Delta_{s,t,u}[f;x,y,z]|\leq M\cdot(\delta(\epsilon))^{-2}(z-u)^2$

In case (iii)and in case (iv) we obtain in similar manner:

(2.6) $|\Delta_{s,t,u}[f;x,y,z]|\leq M(\delta(\epsilon))^{-2}(y-t)^2$

(2.7) $|\Delta_{s,t,u}[f;x,y,z]|\leq M\cdot(\delta(\epsilon))^{-2}(x-s)^2$

Using (2.4),the second and the third inequality of(v),we find that:

(2.8) $|\Delta_{s,t,u}[f;x,y,z]|\le M(\delta(\epsilon))^{-4}(y-t)^2(z-u)^2$

Similary, in the situations (vi) and (vii)we obtain:

(2.9) $|\Delta_{s,t,u}[f;x,y,z]|\le M(\delta(\epsilon))^{-4}(x-s)^2(z-u)^2$

(2.10) $|\Delta_{s,t,u}[f;x,y,z]|\le M(\delta(\epsilon))^{-4}(x-s)^2(y-t)^2$

From the three inequalities of (viii)we find:

(2.11) $|\Delta_{s,t,u}[f;x,y,u]|\le M(\delta(\epsilon))^{-6}(x-s)^2(y-t)^2(z-u)^2$

Employing (2.3),(2.5)-(2.11)we deduce the following inequality:

(2.12)

$$|\Delta_{s,t,u}[f;x,y,z]|\le\frac{\epsilon}{4}+M(\delta(\epsilon))^{-2}(x-s)^2+M(\delta(\epsilon))^{2}(y-t)^2+M(\delta(\epsilon))^{-2}(z-u)^2+$$

$$+M(\delta(\epsilon))^{-4}(x-s)^2(y-t)^2+M(\delta(\epsilon))^{-4}(x-s)^2+M(\delta(\epsilon))^{-4}(y-t)^2(z-u)^2+$$

$$+M(\delta(\epsilon))^{-6}(x-s)^2(y-t)^2(z-u)^2$$

Because $y,t\in[0,1]$, it follows that $(y-t)^2\le 1$. Similary we can write $(x-s)^2\le 1$, $(z-u)^2\le 1$. From (2.12)we conclude that the following inequality:

(2.13) $|\Delta_{s,t,u}[f;x,y,z]|\le\frac{\epsilon}{4}+$

$$+M(\delta(\epsilon))^{-2}\{1+(\delta(\epsilon))^{2}+(\delta(\epsilon))^{-6}\}(x-s)^2+$$

$$+M(\delta(\epsilon))^{-2}\{1+(\delta(\epsilon))^{-2}\}(y-t)^2+$$

$$+M(\delta(\epsilon))^{-2}(z-u)^2$$

holds and the lemma is proved with:

$$A(\epsilon)=M(\delta(\epsilon))^{-2}\{1+2(\delta(\epsilon))^{-2}+(\delta(\epsilon))^{-4}\}$$

$$B(\epsilon)=M(\delta(\epsilon))^{-2}\{1+(\delta(\epsilon))^{-2}\}$$

$$C(\epsilon)=M(\delta(\epsilon))^{-2}$$

Now we shall prove the main result of this paper.Let us consider the following real - valued functions on I^3:

$e_0(s,t,u)=1,\ e_1(s,t,u)=s,\ e_2(s,t,u)=t,\ e_3(s,t,u)=u$

Theorem 2.1: Let *$(L_{m,n,p})_{(m,n,p)\in N^3}$ be a sequence of positive linear operators mapping the functions of $\mathbb{R}^{I^3}$ into the functions of $\mathbb{R}^{I^3}$*

such that all $(x,y,z)\in I^3$ *we have:*

(i) $L_{m,n,p}(e_0;x,y,z)=1$

(ii) $L_{m,n,p}(e_1;x,y,z)=x+\alpha_{m,n,p}(x,y,z)$

(iii) $L_{m,n,p}(e_2;x,y,z)=y+\beta_{m,n,p}(x,y,z)$

(iv) $L_{m,n,p}(e_3;x,y,z)=z+\gamma_{m,n,p}(x,y,z)$

(v) $L_{m,n,p}(e_1^2+e_2^2+e_3^2;x,y,z)=x^2+y^2+z^2+\delta_{m,n,p}(x,y,z)$

where $\{\alpha_{m,n,p}(x,y,z)\},\{\beta_{m,n,p}(x,y,z)\},\{\gamma_{m,n,p}(x,y,z)\},\{\delta_{m,n,p}(x,y,z)\}$ *converge to zero uniformly on* I^3 *as* m,n,p *tend towards to infinity.*

If $f(\cdot,*,\circ)\in B(I^3)$ and $(x,y,z)\in I^3$, we introduce the notation:

(*) $U_{m,n,p}(f;x,y,z)=L(f(\cdot,y,z)+f(x,*,z)+f(x,y,\circ)-f(\cdot,*,z)-$

$-f(\cdot,y,\circ)-f(x,*,\circ)+f(\cdot,*,\circ))$

In these conditions, for every $f\in C_b(I^3)$, the sequence $\{U_{m,n,p}(f)\}$ converges uniformly to f on I^3.

Proof:Let be $(x,y,z)\in I^3$ a fined point.We define the function

$F:I^3\to\mathbb{R}$ by the equality:

$F(\cdot,*,\circ)=f(\cdot,y,z)+f(x,*,z)+f(x,y,\circ)-$

$-f(\cdot,*,z)-f(x,*,\circ)-f(\cdot,*,z)+f(\cdot,*,\circ)$

It is easy to see that:

$$\Delta_{s,t,v}[F;a,b,c]=-\Delta_{s,t,u}[f;a,b,c]$$

From this fact, the B-continuity of f implies that the function F is also B-continuous.

Hence $U_{m,n,p}$ is a well - defined linear operator on $C_b(I^3)$.

Let be $f\in C_b(I^3)$ arbitrarly chosen and let be $(x,y,z)\in I^3$ and $\varepsilon>0$ given.

Because $L_{m,n,p}$ is a linear operator reproducing the constant functions (see the condition(i)),we have:

(2.8) $|\Delta_{s,t,u}[f;x,y,z]|\leq M(\delta(\epsilon))^{-4}(y-t)^2(z-u)^2$

Similary, in the situations (vi) and (vii)we obtain:

(2.9) $|\Delta_{s,t,u}[f;x,y,z]|\leq M(\delta(\epsilon))^{-4}(x-s)^2(z-u)^2$

(2.10) $|\Delta_{s,t,u}[f;x,y,z]|\leq M(\delta(\epsilon))^{-4}(x-s)^2(y-t)^2$

From the three inequalities of (viii)we find:

(2.11) $|\Delta_{s,t,u}[f;x,y,u]|\leq M(\delta(\epsilon))^{-6}(x-s)^2(y-t)^2(z-u)^2$

Employing (2.3),(2.5)-(2.11)we deduce the following inequality:
(2.12)

$$|\Delta_{s,t,u}[f;x,y,z]|\leq\frac{\epsilon}{4}+M(\delta(\epsilon))^{-2}(x-s)^2+M(\delta(\epsilon))^2(y-t)^2+M(\delta(\epsilon))^{-2}(z-u)^2+$$

$$+M(\delta(\epsilon))^{-4}(x-s)^2(y-t)^2+M(\delta(\epsilon))^{-4}(x-s)^2+M(\delta(\epsilon))^{-4}(y-t)^2(z-u)^2+$$

$$+M(\delta(\epsilon))^{-6}(x-s)^2(y-t)^2(z-u)^2$$

Because $y,t\in[0,1]$, it follows that $(y-t)^2\leq 1$. Similary we can write $(x-s)^2\leq 1$, $(z-u)^2\leq 1$. From (2.12)we conclude that the following inequality:

(2.13) $|\Delta_{s,t,u}[f;x,y,z]|\leq\frac{\epsilon}{4}+$

$$+M(\delta(\epsilon))^{-2}\{1+(\delta(\epsilon))^2+(\delta(\epsilon))^{-4}\}(x-s)^2+$$

$$+M(\delta(\epsilon))^{-2}\{1+(\delta(\epsilon))^{-2}\}(y-t)^2+$$

$$+M(\delta(\epsilon))^{-2}(z-u)^2$$

holds and the lemma is proved with:

$$A(\epsilon)=M(\delta(\epsilon))^{-2}\{1+2(\delta(\epsilon))^{-2}+(\delta(\epsilon))^{-4}\}$$

$$B(\epsilon)=M(\delta(\epsilon))^{-2}\{1+(\delta(\epsilon))^{-2}\}$$

$$C(\epsilon)=M(\delta(\epsilon))^{-2}$$

Now we shall prove the main result of this paper.Let us consider the following real - valued functions on I^3:

$e_0(s,t,u)=1$, $e_1(s,t,u)=s$, $e_2(s,t,u)=t$, $e_3(s,t,u)=u$

Theorem 2.1: *Let $(L_{m,n,p})_{(m,n,p)\in N^3}$ be a sequence of positive linear operators mapping the functions of $\mathbb{R}^{I^3}$ into the functions of $\mathbb{R}^{I^3}$*

such that all $(x,y,z)\in I^3$ *we have:*

(i) $L_{m,n,p}(e_0;x,y,z)=1$

(ii) $L_{m,n,p}(e_1;x,y,z)=x+\alpha_{m,n,p}(x,y,z)$

(iii) $L_{m,n,p}(e_2;x,y,z)=y+\beta_{m,n,p}(x,y,z)$

(iv) $L_{m,n,p}(e_3;x,y,z)=z+\gamma_{m,n,p}(x,y,z)$

(v) $L_{m,n,p}(e_1^2+e_2^2+e_3^2;x,y,z)=x^2+y^2+z^2+\delta_{m,n,p}(x,y,z)$

where $\{\alpha_{m,n,p}(x,y,z)\},\{\beta_{m,n,p}(x,y,z)\},\{\gamma_{m,n,p}(x,y,z)\},\{\delta_{m,n,p}(x,y,z)\}$ *converge to zero uniformly on* I^3 *as* m,n,p *tend towards to infinity.*

If $f(\cdot,*,\circ)\in B(I^3)$ and $(x,y,z)\in I^3$, we introduce the notation:

(*) $U_{m,n,p}(f;x,y,z)=L(f(\cdot,y,z)+f(x,*,z)+f(x,y,\circ)-f(\cdot,*,z)-$

$-f(\cdot,y,\circ)-f(x,*,\circ)+f(\cdot,*,\circ))$

In these conditions, for every $f\in C_b(I^3)$, the sequence $\{U_{m,n,p}(f)\}$ converges uniformly to f on I^3.

Proof:Let be $(x,y,z)\in I^3$ a fined point.We define the function $F:I^3\to\mathbb{R}$ by the equality:

$F(\cdot,*,\circ)=f(\cdot,y,z)+f(x,*,z)+f(x,y,\circ)-$

$-f(\cdot,*,z)-f(x,*,\circ)-f(\cdot,*,z)+f(\cdot,*,\circ)$

It is easy to see that:

$$\Delta_{s,t,u}[F;a,b,c]=-\Delta_{s,t,u}[f;a,b,c]$$

From this fact, the B-continuity of f implies that the function F is also B-continuous.

Hence $U_{m,n,p}$ is a well - defined linear operator on $C_b(I^3)$.

Let be $f\in C_b(I^3)$ arbitrarly chosen and let be $(x,y,z)\in I^3$ and $\epsilon>0$ given.

Because $L_{m,n,p}$ is a linear operator reproducing the constant functions (see the condition(i)),we have:

(2.14) $f(x,yz)-U_{m,n,p}(f;x,y,z)=L_{m,n,p}(\Delta_{\cdot,\cdot,\cdot}[f;x,y,z])$

Because $L_{m,n,p}$ is a positive operator,we have:

(2.15) $|L_{m,n,p}(g;x,y,z)|=\max\{L_{m,n,p}(g;x,y,z),L_{m,n,p}(-g;x,y,z)\}$

for every function $g\in C_b(I^3)$

Applying this equality to $g(s,t,u)=\Delta_{s,t,u}[f;x,y,z]$

and,further,using the monotonicity of $L_{m,n,p}$ and the lemma 2.1,we find (with $D(\epsilon)=\max(A(\epsilon),B(\epsilon),C(\epsilon))$)the inequality:

(2.16) $|f(x,y,z)-U_{m,n,p}(f;x,y,z)|\leq$

$$<L_{m,n,p}\left[\frac{\epsilon}{4}+D(\epsilon)((x-\cdot)^2+(y-*)^2+(z-*)^2;x,y,z)\right]$$

After some transformations of (2.16)we arrive to the inequality:

(2.17) $|f(x,y,z)-U_{m,n,p}(f;x,y,z)|\leq$

$$\leq\frac{\epsilon}{4}+D(\epsilon)\cdot L_{m,n,p}(e_1^2+e_2^2+e_3^2;x,y,z)-$$

$$-2D(\epsilon)(xL_{m,n,p}(e_1;x,y,z)+yL_{m,n,p}(e_2;x,y,z)+zL_{m,n,p}(e_3;x,y,z))+$$

$$+D(\epsilon)\cdot(x^2+y^2+z^2)L_{m,n,p}(e_0;x,y,z)$$

Using now the hypotheses (i) through (iv), we can write:

(2.18) $|f(x,y,z)-U_{m,n,p}(f;x,y,z)|\leq$

$$\leq\frac{\epsilon}{4}+D(\epsilon)(\delta_{m,n,p}(x,y,z)-2x\alpha_{m,n,p}(x,y,z)-2y\beta_{m,n,p}(x,y,z)-2z\gamma_{m,n,p}(x,y,z))$$

Letting m,n,p to tend to infinity we arrive at the desired result.

Remark 2.1. If the hypothesis (i) of the theorem does not hold,then the equality (2.14)it not true.If one replace (i) by:

(i') $L_{m,n,p}(f;x,yz)=1+V_{m,n,p}(x,y,z)$,

then it is easy to see that the following inequality holds:

(2.19) $|f(x,y,z)-U_{m,n,p}(f;x,y,z)|\leq$

$$\leq |f(x,y,z)|\cdot|V_{m,n,p}(x,y,z)|+\frac{\epsilon}{4}(1+V_{m,n,p}(x,y,z))+$$

$$+D(\epsilon)(\delta_{m,n,p}(x,y,z)-2x\alpha_{m,n,p}(x,y,z)-2y\beta_{m,n,p}(x,y,z)-2z\gamma_{m,n,p}(x,y,z))$$

For $V_{m,n,p}(x,y,z)=0$, it reduces to the inequality (2.18).

The inequality (2.19)allows us to see that we have only pointwise convergence to f(x,y,z), for all $(x,y,z)\in I^3$,if $\{V_{m,n,p}(x,y,z)\}$ converges uniformly to zero as m,n,p tend to infinity.

At the end of this section, we mention that the results of lemma 2.1, theorem 2.1 and remark 2.1 represent the analogues in $C_b(I^3)$of the results given in [1].

3.APPLICATIONS

We consider the case in wich the operator $L_{m,n,p}$from theorem 2.1is the product of the parametrical extensions of the three positive linear operators $L_m, \overline{L_n}, \overline{\overline{L_p}}:\mathbb{R}^I\to\mathbb{R}^I$, given by:

$$L_m(f;x)=)\sum_{i=0}^{m} f(x_i)p_{m,i}(x),\ x_i\in I,\ p_{m,i}(x)\geq 0 \text{ for } 0\leq i\leq m \text{ and all } x\in I$$

$$\overline{L_n}(g;y)=\sum_{j=0}^{n} g(y_j)q_{n,j}(y),\ y_j\in I,\ q_{n,j}(y)\geq 0 \text{ for } 0\leq j\leq n \text{ and all } y\in J$$

$$\overline{\overline{L_p}}(h,z)=\sum_{k=0}^{p} h(z_k)l_{p,k}(z),\ z_k\in I,\ l_{p,k}(z)\geq 0 \text{ for } 0\leq k\leq p \text{ and all } z\in I$$

We assume that:

$$\sum_{i=0}^{m} p_{m,i}(x)=\sum_{j=0}^{n} q_{n,j}(y)=\sum_{k=0}^{p} l_{p,k}(z)=1, \text{ for all } x,y,z\in I$$

Let us denote by $:{}_xL_m, {}_y\overline{L_n}, {}_z\overline{\overline{L_p}}$ the parametrical extensions of $L_m, \overline{L_n}, \overline{\overline{L_p}}$. One observes that:

$$L_{m,n,p}(f;x,y,z)=({}_xL_m\circ{}_y\overline{L_n}\circ{}_z\overline{\overline{L_p}})(f;x,y,z)=$$

$$=\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p} f(x_i,y_j,z_k)\,p_{m,i}(x)\,q_{n,j}(y)\,l_{p,k}(z)$$

is a linear positive operator defined for any $f\in\mathbb{R}^{I^3}$.

Let us assume that the positive linear operator L_m satisfyes the conditions:

$L_m(e_1;x)=x+u_m(x)$ for all $x\in I$

$L_m(e_2;x)=x^2+w_m(x)$ for all $x\in I$

where $e_i(x)=x^i$ $(i=1,2)$. Similarly, we assume that $\overline{L_n},\overline{\overline{L_p}}$ satisfy similar conditions, with $u_m(x),w_m(x)$ replaced by $u_n'(y),w_n'(y)$, respectively by $u_p''(z),w_p''(z)$. It follows that we can write:

$$L_{m,n,p}(e;x,y,z)=1 \qquad (I(s,t,u)=1)$$

$$L_{m,n,p}(\varphi;x,y,z)=x+u_m(x)=x+\alpha_{m,n,p}(x,y,z) \qquad (\varphi(s,t,u)=s)$$

$$L_{m,n,p}(\psi;x,y,z)=y+u_n'(y)=y+\beta_{m,n,p}(x,y,z) \qquad (\psi(s,t,u)=t)$$

$$L_{m,n,p}(\theta;x,y,z)=z+u_p''(z)=z+\gamma_{m,n,p}(x,y,z) \qquad (\theta(s,t,u)=u)$$

$$L_{m,n,p}(\varphi^2+\psi^2+\theta^2;x,y,z)=x^2+y^2+z^2+w_m(x)+w_n'(y)+w_p''(z)=$$

$$=x^2+y^2+z^2+\delta_{m,n,p}(x,y,z)$$

If $(u_m(x))_{m\in N}$, $(u_n'(y))_{n\in N}$, $(u_p''(z))_{p\in N}$, $(w_m(x))_{m\in N}$, $(w_n'(y))_{n\in N}$, $(w_p''(z))_{p\in N}$ converge uniformly to zero as m,n,p tend to infinity, then $\{\alpha_{m,n,p}(x,y,z)\}$ $\{\beta_{m,n,p}(x,y,z)\}$, $\{\gamma_{m,n,p}(x,y,z)\}$, $\{\delta_{m,n,p}(x,y,z)\}$ converges to zero uniformly on I^3, as m,n,p tend to infinity.

Applying the theorem 2.1 and the observations from above, we can state:

Theorem 3.1: *If the sequence of linear and positive operators* $(L_m)_{m\in N}$, $(\overline{L_n})_{n\in N}$, $(\overline{\overline{L_p}})_{p\in N}$ *are given as above and if*

$L_m e_i \to e_i, \overline{L}_n e_i' \to e_i', \overline{\overline{L}}_p e_i'' \to e_i''$, *uniformly on I, for* $i \in \{1,2\}$, *then the operators* $U_{m,n,p}$, *defined by (*), have the property that* $\{U_{m,n,p}(f)\}_{(m,n,p)\in N^3}$ *converges uniformly to f for each* $f \in C_b(I^3)$ *as m,n,p tend to infinity.*

We shall present some applications of theorem 3.1.

Example 3.1:Bernstein - Stancu type operators.

We consider the case of Berustein - Stancu type operators $B_m^{[\alpha]}, \overline{B}_n^{[\beta]}, \overline{\overline{B}}_p^{[\gamma]}; \mathbb{R}^I \to \mathbb{R}^I$ given by:

$$B_m^{[\alpha]}(f;x) = \sum_{i=0}^{m} f\left(\frac{i}{m}\right) w_{m,i}(x,\alpha) \quad \text{for all } x \in I$$

$$\overline{B}_n^{[\beta]}(g;y) = \sum_{j=0}^{n} g\left(\frac{j}{n}\right) \overline{w}_{n,j}(y,\beta) \quad \text{for all } y \in I$$

$$\overline{\overline{B}}_p^{[\gamma]}(h;z) = \sum_{k=0}^{p} h\left(\frac{k}{p}\right) \overline{\overline{w}}_{p,k}(z,\gamma) \quad \text{for all } z \in I$$

where:

$$w_{m,i}(x,\alpha) = \binom{m}{i} \frac{x^{[i,-\alpha]}(1-x)^{[m-i,-\alpha]}}{1^{[m,-\alpha]}}$$

$$\overline{w_{n,j}}(y,\beta) = \binom{n}{j} \frac{y^{[j,-\beta]}(1-y)^{[n-j,-\beta]}}{1^{[n,-\beta]}}$$

$$\overline{\overline{w}}_{p,k}(z,\gamma) = \binom{p}{k} \frac{z^{[k,-\gamma]}(1-z)^{[p-k,-\gamma]}}{1^{[p,-\gamma]}}$$

while $\alpha = \alpha(m) \ge 0, \beta = \beta(n) \ge 0, \gamma = \gamma(n) \ge 0$ and $u^{[n,h]}$ is the factorial power of u, with the increment h.

The functions $w_{m,i}(x,\alpha), \overline{w}_{n,j}(y,\beta), \overline{\overline{w}}_{p,k}(z,\gamma)$ satisfay the following equalities (see[11],[12]):

$$\sum_{i=0}^{m} w_{m,i}(x,\alpha) = \sum_{j=0}^{n} \overline{w}_{n,j}(y,\beta) = \sum_{k=0}^{p} \overline{\overline{w}}_{p,k}(z,\gamma) = 1 \quad \text{for each } x,y,z \in I$$

If ${}_xB_m^{[\alpha]}, {}_y\overline{B}_n^{[\beta]}, {}_z\overline{\overline{B}}_p^{[\gamma]}$ are the parametrical extensions of $B_m^{[\alpha]}, \overline{B}_n^{[\beta]}, \overline{\overline{B}}_p^{[\gamma]}$, we find that:

$$B_{m,n,p}^{[\alpha,\beta,\gamma]}(f;x,y,z)=({}_xB_m^{[\alpha]}\circ{}_y\overline{B}_n^{[\beta]}\circ{}_z\overline{\overline{B}}_p^{[\gamma]})(f;x,y,z)=$$

$$=\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p}f\left(\frac{i}{m},\frac{j}{n},\frac{k}{p}\right)w_{m,i}(x,\alpha)\overline{w}_{n,j}(y,\beta)\overline{\overline{w}}_{p,k}(z,\gamma)$$

is a positive linear operator, defined for any function $f\in\mathbb{R}^{I^3}$.

Denoting by $P_{m,n,p}^{[\alpha,\beta,\gamma]}$ the operator given in [3],we have:

$$P_{m,n,p}^{[\alpha,\beta,\gamma]}={}_xB_m^{[\alpha]}\oplus{}_y\overline{B}_n^{[\beta]}\oplus{}_z\overline{\overline{B}}_p^{[\gamma]}$$

The operators $B_m^{[\alpha]}, \overline{B}_n^{[\beta]}, \overline{\overline{B}}_p^{[\gamma]}$ have the following properties (see [11],[12]):

$$B_m^{[\alpha]}(e_1;x)=x,\ \overline{B}_n^{[\beta]}(e_1';y)=y,\ \overline{\overline{B}}_p^{[\gamma]}(e_1'';z)=z$$

$$B_m^{[\alpha]}(e_2;x)=\frac{1}{1+\alpha}\left[\frac{x(1-x)}{m}+x(x+\alpha)\right]$$

$$\overline{B}_n^{[\beta]}(e_2';y)=\frac{1}{1+\beta}\left[\frac{y(1-y)}{n}+y(y+\beta)\right]$$

$$\overline{\overline{B}}_p^{[\gamma]}(e_2'';z)=\frac{1}{1+\gamma}\left[\frac{z(1-z)}{p}+z(z+\gamma)\right]$$

It is easy to see that if $\alpha=\alpha(m),\beta=\beta(n),\gamma=\gamma(n)$ tend to zero as m,n,p tend to infinity then

$B_m^{[\alpha]}(e_i)\to e_i, B_n^{[\beta]}(e_i)\to e_i', \overline{\overline{B}}_p^{[\gamma]}(e_i'')$,uniformly on I,for $i\in\{1,2\}$.

We can formulate the result given in [3] as a consequence of theorem 3.1,by:

Corollary 3.1: If $\alpha=\alpha(m),\beta=\beta(n),\gamma=\gamma(p)$ tend to zero as m,n,p tend to infinity,then the sequence $\{P_{m,n,p}^{[\alpha,\beta,\gamma]}\}(m,n,p)\in\mathbb{N}^3$, defined by:

$$P_{m,n,p}^{[\alpha,\beta,\gamma]}(f;x,y,z)=\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p}w_{m,i}(x,\alpha)\overline{w}_{n,j}(y,\beta)\overline{\overline{w}}_{p,k}(z,\gamma)\cdot$$

$$\cdot[f(\frac{i}{m},y,z)+(x,\frac{j}{n},z)+f(x,y,\frac{k}{p})-f(\frac{i}{m},\frac{j}{n},z)-f(\frac{i}{m},y,\frac{k}{p})\cdot$$

$$-f(x,\frac{j}{n},\frac{k}{p})+f(\frac{i}{m},\frac{j}{n},\frac{k}{p})],$$

converges uniformly to f, for any $f\in C_b(I^3)$.

Example 3.2:A Bleiman - Butzer - Hahn type operator.
In [4],B.Bleiman, P.L.Butzer and L.Hahn have considered the positive linear operator L_m, defined by:

$$L_m(f;x)=\frac{1}{(1+x)^m}\sum_{i=0}^{m}f\left(\frac{i}{m-i+1}\right)\binom{m}{i}x^i,\quad m\in\mathbb{N}$$

for any continuous function $f:[0,+\infty)\to\mathbb{R}$
In [2],by using the operator L_m,we have constructed a linear operator $L_{m,n}$,defined for any function $f\in C_b(I^2)$,and we have proved that rhe sequence $\{L_{m,n}f\}_{(m,n)\in\mathbb{N}^2}$, converges uniformly to f, for any function $f\in C_b(I^2)$

Now we consider the operator $L_m,\overline{L}_n,\overline{\overline{L}}_p:\mathbb{R}^I\to\mathbb{R}^I$, defined by:

$$L_m(f;x)=\sum_{i=0}^{m}f\left(\frac{i}{m-i+1}\right)P_{m,i}(x),\quad P_{m,i}(x)=\frac{x^i}{(1+x)^m}\binom{m}{i}$$

$$\overline{L}_n(g;y)=\sum_{j=0}^{n}g\left(\frac{j}{n-j+1}\right)\overline{q}_{n,j}(y),\quad \overline{q}_{n,j}(y)=\frac{y^j}{(1+y)^n}\binom{n}{j}$$

$$\overline{\overline{L}}_p(h;z)=\sum_{k=0}^{p}h\left(\frac{k}{p-k+1}\right)\overline{\overline{r}}_{p,k}(z),\quad \overline{\overline{r}}_{p,k}(z)=\frac{z^k}{(1+z)^p}\binom{p}{k}$$

It is easy to see that:

$$\sum_{i=0}^{m}P_{m,i}(x)=\sum_{j=0}^{n}\overline{q}_{n,j}(y)=\sum_{k=0}^{p}\overline{\overline{r}}_{p,k}(z)=1.$$

for all $x,y,z\in I=[0,1]$. $x,y,z\in I=[0,1]$

From [4], we know that $L_m(e_i)\to e_i$, $\overline{L}_n(e_i')\to e_i'$ $\overline{\overline{L}}_p(e_i'')\to e_i''$ $(i\in\{1,2\})$, uniformly on I as m,n,p tend to infinity.

Denoting by : ${}_xL_m, {}_y\overline{L}_n, {}_z\overline{\overline{L}}_p$ the parametrical extensions of $L_m, \overline{L}_n, \overline{\overline{L}}_p$, we obtain that:

$$U_{m,n,p}(f;x,y,z)=({}_xL_m\circ{}_y\overline{L}\circ\overline{\overline{L}}_p)(f;x,y,z)=$$

$$=\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p}p_{m,i}(x)\overline{q}_{n,j}(y)\overline{\overline{r}}_{p,k}(z)f(\frac{i}{m},\frac{j}{n},\frac{k}{p}),$$

is a linear positive operator, defined for any $f\in\mathbf{R}^{I^3}$.

We consider nov the operator $L_{m,n,p}={}_xL_m\oplus{}_y\overline{L}_n\oplus{}_z\overline{\overline{L}}_p$,

$$L_{m,n,p}(f;x,y,z)=\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p}p_{m,i}(x)\overline{q}_{n,j}(y)\overline{\overline{r}}_{p,k}(z)\cdot$$

$$\cdot[f(\frac{i}{m},y,z)+f(x,\frac{j}{n},z)+f(x,y,\frac{k}{p})-f(\frac{i}{m},\frac{j}{n},z)-f(\frac{i}{m},y,\frac{k}{p})-$$

$$-f(x,\frac{j}{n},\frac{k}{p})+f(\frac{i}{m},\frac{j}{n},\frac{k}{p})]$$

for any function $f\in C_b(I^3)$.

As a consequence of the theorem 3.1, we have the following:

Corollary 3.2: The sequence $\{L_{m,n,p}(f;x,y,z)\}_{(m,n,p)\in N^3}$ converges uniformly to f as m,n,p tend to infinity, for any function $f\in C_b(I^3)$

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