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GENERALIZED CONTRACTIONS AND HIGHER ORDER HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We establish existence and uniqueness results of (periodic) solutions for higher order partial differential equations by means of a generalized contraction principle.

A generalized Lipschitz condition (or Perron condition) is assumed instead of the classical Lipschitz condition.

1. INTRODUCTION

In a recent paper [15], Sheng and Agarwal gave existence and uniqueness results for higher order nonlinear hyperbolic partial differential equations. Their paper extend the work of Aziz [3] and Cesari [10-12], devoted to the study of periodic solutions of certain second order nonlinear hyperbolic partial differential equations. As Aziz in [3], Sheng and Agarwal consider a generalized Lipschitz condition and, by means of the well-k nown Schauder's and Banach's fixed point theorems, they establish the existence and the uniqueness of the solutions, respectively.

Motivated with our recent works for second order partial differential equations [7], integral equations of Fredholm type [6] or Volterra type [9] and infinite system of ordinary differential equations [8] where generalized fixed point argument, based on K-valued norms is considered, in this paper, we shall generalise the Sheng and Agarwal's result concerning the uniqueness of solutions. The existence part in the Sheng and Agarval will be treated in a future work.

A strong tool in obtaining existence and uniqueness results for differential or integral equations is the contraction mapping principle (or the Banach's, fixed point theorem). To apply this principle to concrete problems we construct an associate operator so that the initial equation is equivalent to a fixed point problem for the obtained operator. Next we have to find a space where the operator in question is contractive, that is to construct a norm, equivalent to the norms of the spaces into consideration, with respect to which the operator is contractive.

As in [6]-[9], in the present paper we shall use an alternative of the above mentioned approach, i.e. we shall obtain existence and uniqueness theorems by means of a generalized contraction principle [5], when generalized norms, with values in the positive cone of a real Banach space, rather equivalent scalar norms are considered.

2. THE GENERALIZED CONTRACTION MAPPING PRINCIPLE

In this section we shall recall, in an easily modified statement, the generalized contraction mapping principle given in the paper [5] (see also [13]-[14]).

Let $(E, \|\cdot\|)$ be a real Banach space. A set $K \subset E$ is called a cone if

(i) K is closed

(ii) $x, y \in K$ implies $ax + by \in K$ for all $a, b \in \mathbb{R}_+$;

(iii) $K \cap (-K) = \{\theta\}$, where θ is the null element of E .

The cone K induces a reflexive, transitive and antisymmetrical order relation $\le$ in E, by

$$x \le y \quad \text{if and only if} \quad y-x \in K,$$

related to the linear structure by the properties

$$u \le v \quad \text{implies} \quad u+z \le v+z, \quad \text{for each} \quad z \in \mathbb{E}$$

and

$$u \le v \quad \text{implies} \quad tu \le tv, \quad \text{for each} \quad t \in \mathbb{R}_+,$$

that is "$\le$" is a linear order relation.

The space E endowed with this order relation is called an **ordered Banach space**, while K is termed as its **positive cone.**

We say that the norm of E is **monotone** if

$$x,y \in \mathbb{E},\ \theta \le x \le y \quad \text{implies} \quad \|x\| \le \|y\|.$$

The cone K is **normal** if there exists $\delta > 0$ such that, from

$$x,y \in \mathbb{E},\ x \ge 0,\ y \ge 0 \quad \text{and} \quad \|x\| = \|y\| = 1$$

it results

$$\|x+y\| \ge \delta.$$

Recall that if the norm of $(\mathbb{E}, \|\cdot\|)$ is monotone, then K is a normal cone (see [7] with the references therein).

Throughout this paper K will be the positive cone in a real ordered Banach space $(\mathbb{E}, \|\cdot\|)$ with monotone norm.

We need some definitions and results from [7].

DEFINITION 2.1 A mapping $\omega : K \to K$ is a **comparison function** if

(i) ω is nondecreasing;

(ii) $(\omega^n(t))_{n \in \mathbb{N}}$ converges to θ, for all $t \in K$.

Example 2.1 Let $\mathbb{E} = \mathbb{R}$, the real axis, with the usual norm.

In this case $K = \mathbb{R}_+$ and a usual comparison function is

$\omega : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\omega(t) = at,\ 0 \le a < 1,\ t \in \mathbb{R}_+.$$

DEFINITION 2.2 A mapping $\omega : K \to K$ is called **(c)-comparison function** if ω is nondecreasing and fulfils the following convergence condition

(c) There exist two numbers $k_0, \alpha, 0 \le \alpha < 1$, and a convergent series of nonnegative real terms $\sum_{k=1}^{\infty} a_k$ such that

$$\|\omega^{k+1}(t)\| \le \alpha \|\omega^k(t)\| + a_k, \quad \text{for} \quad k \ge k_0, \ (\forall)\ t \in K. \tag{2.1}$$

Remarks. 1) In [7] condition (2.1) appears in a different but equivalent form ;

2) Every (c)-comparison function is a comparison function. Indeed, if ω is a (c)-comparison function, then the series $\sum_{k=1}^{\infty} \|\omega^k(t)\|$ converges for all $t \in K$ and, consequently, the series

$$\sum_{k=1}^{\infty} \omega^k(t) \tag{2.2}$$

converges in E, that is condition (ii) in Definition 2.1 is satisfied.

3) The function ω from Example 2.1 is actually a (c)-comparison function.

DEFINITION 2.3. Let X be a nonemty set. A mapping $d: X \times X \to K$ is said to be a K-metric on X if

(i) $d(x,y) \ge \theta$ and $d(x,y) = \theta$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$, for all $x, y \in X$;

(iii) $d(x,y) \le d(x,z) + d(y,z)$, for all $x, y, z \in X$.

The set X endowed with a K-metric d is called **K-metric space**, denoted as usually for metric spaces by (X,d).

DEFINITION 2.4. Let (X,d) be a K-metric space. A mapping $f: X \to X$ is called **abstract ω-contraction** or **ω-contraction** if there exists a comparison function $\omega: K \to K$ such that

$$d(f(x),f(y)) \le \omega(d(x,y)),\ x,y\in X. \tag{2.3}$$

Example 2.2. Let $K=\mathbb{R}_+$ and let ω be the comparison function in Example 2.1. Then a $\mathbb{R}_+$-metric space is an usual metric space, while condition (2.3) becomes

$$d(f(x),f(y)) \le a\cdot d(x,y),\ 0\le a<1,\ x,y\in X\ ,$$

the well-known contraction condition in the Banach's contraction mapping principle.

Remark. In a K-metric space all concepts as K-fundamental sequence, K-convergent sequence, complete K-metric space and so on, are defined in a similar way to that for the usual metric space.

The following theorem is a natural generalization of the contraction mapping principle. To state it we need some additional remarks and results adapted from [4]

LEMMA 2.1 *[16]. Let $\omega:K\to K$ be a (c)- comparison function.*

If $\sigma:K\to K$ is given by

$$\sigma(t)=\sum_{k=1}^{\infty}\omega^k(t),\ t\in K$$

then σ is continuous at θ and nondecreasing.

LEMMA 2.2 *([16]). If $\omega:K\to K$ is a subadditive comparison function, then ω is continuous.*

THEOREM 2.1. *Let (X,d) be a complete K-metric space, with K a normal cone and let $f:X\to X$ be a ω-contraction, with ω a (c)-comparison function.*
Then

(1) $F_f=\{x^*\}$, where $F_f=\{x\in X \mid f(x)=x\}$;

(2) The sequence $(x_n)_{n\in\mathbb{N}}$, $x_n=f(x_{n-1})$, $n\ge1$ converges to x^*, for every $x_0\in X$;

(3) We have

$$d(x_n,x^*)\le\sigma(d(x_n,x_{n+1})),\ n\ge0; \tag{2.4}$$

(4) If, in addition, ω is subadditive and there exist

$\eta \in K$, $\eta \neq \theta$ *and a mapping* $g: X \to X$ *so that*

$$d(f(x), g(x)) < \eta, \quad \text{for all} \quad x, y \in X,$$

then

$$d(y_n, x^*) \leq \sigma(\eta) + \sigma(\omega^n(d(x_0, x_1))),$$

where

$$y_n = g^n(x_0).$$

Remarks.1) For $K = \mathbb{R}_+$ and ω as in Example 2.1, from Theorem 2.1 we obtain the Banach's fixed point theorem.

In this case $\sigma(t) = \frac{1}{1-a} \cdot t$, hence the estimation (4) gives

$$d(x_n, x^*) \leq \frac{1}{1-a} d(x_n, x_{n+1}), \quad n \geq 0$$

which is a very usefull stopping criterion from a practical point of view. An a priori order estimation in Theorem 2.1 may be obtained from (2.4) as

$$d(x_n, x^*) \leq \sigma(\omega^n(d(x_0, x_1))), \tag{2.5}$$

which in the above particular case becomes

$$d(x_n, x^*) \leq \frac{a^n}{1-a} d(x_0, x_1). \tag{2.5'}$$

2)A local version of Theorem 2.1 may be easily stated and proved. We give it here because we obtain as a particular case Theorem 3.1 in the Sheng and Agarwall's paper, which they use in establishing their uniqueness result.

THEOREM 2.2. *Let* (X, d) *be a complete K-metric space. Let* $r \in K$, $r \neq \theta$ *and* $S(u_0, r) = \{ u \in X \mid d(u, u_0) \leq r \}$.

If T *maps* $S(u_0, r)$ *into* X *and there exists a (c)-comparison function* ω *such that*

(i) T *is a* ω*-contraction on* $S(u_0, r)$;

(ii) $d(u_0, Tu_0) \leq r - \omega(r)$.

Then, T *has a unique fixed point* u^* *in* $S(u_0, r)$ *and the sequence* (u_n)

defined by

$$u_{k+1}=Tu_k,\ k=0,1,2,\ldots$$

converges to u^* *with*

$$d(u_k,u^*)\le\sigma(d(x_k,x_{k+1})),\ n\ge1. \tag{2.6}$$

Remarks. 1) For $K=\mathbb{R}_+$, X a Banach space, and ω as in Example 2.1, from Theorem 2.2 we obtain Theorem 3.1 in [15].

2) In the above mentioned particular case from (6) we obtain (here $s=\sigma$) the a posteriori estimation

$$\|u_k-u^*\|\le\frac{1}{1-\alpha}\cdot\|x_k-x_{k+1}\|,$$

which is better than the a priori estimation

$$\|u_k-u^*\|\le\alpha^k r_o,$$

in Theorem 3.1 [15], obtained from (2.5).

3. EXISTENCE AND UNIQUENESS RESULTS

Le a,b be two positive numbers and let us denote $[0,a],[0,b]$ and $[0,a]x[0,b]$ as I_a, I_b, and I_{ab}, respectively.

We shall consider the nonlinear hyperbolic partial differential equation

$$\frac{\partial^{n+m}u}{\partial x^n\partial y^m}=g(x,y,\langle u\rangle),\ n,m\ge1,\ (x,y)\in I_{ab}, \tag{3.1}$$

together with the periodic boundary conditions

$$\frac{\partial^i u(0,y)}{\partial x^i}=\frac{\partial^i u(a,y)}{\partial x^i}=\psi_i(y),\ 0\le i\le n-1,\ y\in I_b, \tag{3.2-1}$$

$$\frac{\partial^j(x,0)}{\partial y^j}=\phi_j(x),\ 0\le j\le m-1,\ x\in I_a, \tag{3.2-2}$$

and

$$\frac{\partial^i u(0,y)}{\partial x^i}=\frac{\partial^i u(a,y)}{\partial x^i}=\psi_i(y),\ 0\le i\le n-1,\ y\in I_b, \tag{3.3-1}$$

$$\frac{\partial^j u(x,0)}{\partial y^j} = \frac{\partial^j u(x,b)}{\partial y^j} = \phi_j(x), \quad 0 \le j \le m-1, \; x \in I_a, \qquad (3.3\text{-}2)$$

respectively, where $\langle u \rangle$ denotes the vector

$$\left(u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^n u}{\partial x^n}, \ldots, \frac{\partial^j u}{\partial y^j}, \ldots, \frac{\partial^{n+j} u}{\partial x^n \partial y^j}, \ldots, \right.$$

$$\left. \ldots, \frac{\partial^{m-1} u}{\partial y^{m-1}}, \ldots, \frac{\partial^{n+m-1} u}{\partial x^n \partial y^{m-1}}, \frac{\partial^m u}{\partial y^m}, \ldots, \frac{\partial^{n+m-1} u}{\partial x^{n-1} \partial y^m} \right)$$

and $g \in C[\, I_{ab} x \mathbb{R}^{nm+n+m}, \mathbb{R} \,]$.

For $n=m=1$ we obtain the second nonlinear hyperbolic partial differential equation

$$u_{xy} = g(x, y, u, u_x, u_y),$$

with the boundary conditions $u(0,y) = \psi(y), u(x,0) = \phi(x)$,

$\phi(0) = \psi(0)$, which have been extensively studied in [3],[10-15]. Cesari showed in [4] that if,in addition to the usual smoothness requirement, g and the boundary function ϕ, are both periodic with respect to x of period T, then there exists a periodic solution of the above problem.

Let us come back to the problems (3.1)+(3.2) and (3.1)+(3.3).

We need necessary conditions to ensure a function $u(x,y)$ to be a solution of (3.1)+(3.2) and (3.1)+(3.3), respectively.

These lemmas are proved in [15].

LEMMA 3.1 *([15])For a function* $u(x,y)$ *to be a solution of (3.1)+(3.2) it is necessary that*

$$\frac{\partial^i u(0,y)}{\partial x^i} = \sum_{k=i}^{n-1} \frac{a^{k-i}}{(k-i)!} \frac{\partial^k u(0,y)}{\partial x^k} + \frac{1}{(n-i-1)!} \int_0^a (a-\xi)^{n-i-1}$$

$$\left(\sum_{k=0}^{m-1} \frac{y^k}{k!} \cdot \frac{\partial^{n+k} u(\xi,0)}{\partial \xi^n \partial y^k} + \frac{1}{(m-1)!} \int_0^y (y-\xi)^{m-1} g(\xi, \zeta \langle u \rangle)\, d\zeta \right) d\xi.$$

$$0 \le i \le n-1 \;\; y \in I_b \qquad (3.4)$$

LEMMA 3.2 *[18]For a function $u(x,y)$ to be solution of (3.1)+(3.3), in addition to(4.4), it is necessary that*

$$\frac{\partial^j u(x,0)}{\partial y^j}=\sum_{k=j}^{m-1}\frac{b^{k-j}}{(k-j)!}\cdot\frac{\partial^k u(x,0)}{\partial y^k}+\frac{1}{(m-j-1)!}\int_0^b (b-\zeta)^{m-j-1}$$

$$\left(\sum_{k=0}^{n-1}\frac{x^k}{k!}\cdot\frac{\partial^{k+m}u(0,\zeta)}{\partial x^k\partial\zeta^m}+\frac{1}{(n-1)!}\int_0^x (x-\xi)^{n-1}g(\xi,\zeta,\langle u\rangle d\xi\right)d\zeta,$$

$$0\le j\le m-1,\ x\in I_a. \tag{3.5}$$

Let now $\theta\in C[I_{ab}]$. We define, for $i,j\ge 1$

$$(B_{i,j}\theta)(x,y)=\frac{1}{(i-1)!(j-1)!}\int_0^x\int_\xi^y (x-\xi)^{i-1}(y-\zeta)^{j-1}\theta(\xi\zeta)\,d\zeta\, d\xi+$$

$$+\sum_{p=1}^{i}\frac{\partial^{n+m-p-j}u(0,y)}{\partial x^{n-p}\,\partial y^{m-j}}\left(\frac{x^{i-p}}{(i-p)!}\right)+$$

$$+\sum_{q=1}^{j}\frac{\partial^{n+m-i-q}u(x,0)}{\partial x^{n-i}\,\partial y^{m-q}}\left(\frac{y^{j-q}}{(j-q)!}\right)-\sum_{p=1}^{i}\sum_{q=1}^{j}\frac{\partial^{n+m-p-q}u(0,0)}{\partial x^{n-p}\,\partial y^{m-q}}$$

$$\cdot\left(\frac{x^{i-p}}{(i-p)!}\right)\left(\frac{y^{j-q}}{(j-q)!}\right),\quad (x,y)\in I_{ab}; \tag{3.6-1}$$

for $j\ge 1$

$$(B_{0,j}\theta)(x,y)=\frac{1}{(j-1)!}\int_0^y (y-\zeta)^{j-1}\,\theta(x,\zeta)\,d\zeta+$$

$$+\sum_{q=1}^{j}\frac{\partial^{n+m-q}u(x,0)}{\partial x^{n}\,\partial y^{m-q}}\left(\frac{y^{j-q}}{(j-q)!}\right),\quad (x,y)\in I_{ab}; \tag{3.6-2}$$

and for $i\ge 1$

$$(B_{i,0}\theta)_{(x,y)}=\frac{1}{(i-1)!}\int_0^x (x-\xi)^{i-1}\,\theta(\xi,y)\,d\xi+$$

$$+\sum_{p=1}^{i}\frac{\partial^{n+m-p}u(0,y)}{\partial x^{n-p}\,\partial y^{m}}\left(\frac{x^{i-p}}{(i-p)!}\right),\quad (x,y)\in I_{ab}; \tag{3.6-3}$$

It is easy to see that if $u(x,y)$ is a solution of problem (3.1)+(3.2) on I_{ab}, then

$$\frac{\partial^{n+m}u}{\partial x^n \partial y^m} = s(x,y) \in C[I_{ab}]$$

satisfies the functional equation

$$s(x,y) = g(x,y,\langle B_{n,m}s\rangle). \qquad (3.7)$$

Further, if $\phi_j \in C^n[I_a]$, $j=0,1,\ldots,m-1$, $\psi_i \in C^m[I_b]$, $i=0,1,\ldots,n-1$, then if we denote

$$t(x,y) = \frac{\partial^{n+m}u}{\partial x^n \partial y^m}, \qquad (3.8)$$

where u is a solution of (3.1)+(3.3) on I_{ab}, then

$$t(x,y) = g(x,y,\langle B_{n,m}t\rangle), \qquad (3.9)$$

and therefore $t(x,y)$ is a fixed point on $C[I_{ab}]$ of the operator T_2, given by

$$(T_2, t)_{(x,y)} = g(x,y,\langle B_{n,m}t\rangle). \qquad (3.10)$$

Let now assume that the following hypotheses hold for problem (3.1)+(3.3):

(a_1) $g(x,y,\langle v\rangle)$ is continuous for x,y and $v_{i,j}$, $0\le i\le n$, $0\le j\le m$, $i+j\le n+m-1$, and periodic in x and y with period a and period b, respectively;

(b_1) $g(x,y,\langle v\rangle)$ is uniformly bounded, i.e. $|g|\le L$;

(c_1) $|g(x,y,\langle v\rangle)-g(x,y,\langle \bar{v}\rangle)|\le h(x,y,|\langle v\rangle-\langle \bar{v}\rangle|)$ $x,y) \in I_{a,b}$

where $h: I_{ab}x\mathbb{R}^{nm+n+m}\to\mathbb{R}$ is nondecreasing with respect to the third argument, that is $v_1\le v_2 \Rightarrow h(x,y,v_1)\le h(x,y,v_2)$;

(d_1) $\phi_j \in C^n(I_a)$, $j=0,1,\ldots,m-1$, $\psi_i \in C^m(I_b)$, $i=0,1,\ldots,n-1$, are periodic in x and y with period a and b, respectively, and

$$\frac{d^i\phi_j(0)}{dx^i}=\frac{d^j\psi_i(0)}{dy^j},\ 0\le i\le n-1,\ 0\le j\le m-1. \qquad (3.11)$$

DEFINITION 3.1.([15]).The functions $\mathcal{B},\hat{\mathcal{E}}\in C(I_{ab})$ are called aproximate solution of (3.7) and (3.9), respectively, if there exist small constants $\delta_i>0,\ i=1,2,$ such that

$$\max_{(x,y)\in I_{ab}}|\mathcal{B}(x,y)-g(x,y,\langle B_{n,m}\mathcal{B}\rangle)|\le\delta_1 \qquad (3.12)$$

and

$$\max_{(x,y)\in I_{ab}}|\hat{\mathcal{E}}(x,y)-g(x,y,\langle B_{n,m}\hat{\mathcal{E}}\rangle)|\le\delta_2 \qquad (3.13)$$

Remark.If $\mathcal{B}$ and $\hat{\mathcal{E}}$ are approximate solutions of (3.7) and (3.9), respectively, then there exist continuous functions $\eta_i(x,y),\ i=1,2,$ such that

$$\mathcal{B}(x,y)=g(x,y,\langle B_{n,m}\mathcal{B}\rangle)+\eta_1(x,y)$$

and

$$\hat{\mathcal{E}}(x,y)=g(x,y,\langle B_{n,m}\hat{\mathcal{E}}\rangle)+\eta_2(x,y),$$

whith

$$\max_{(x,y)\in I_{ab}}|\eta_i(x,y)|\le\delta_i\ \ i=1,2.$$

Let $C(I_{ab})$ be the Banach space endowed with the usual norm

$$\|t\|=\sup_{(x,y)\in I_{ab}}|t(x,y)|,\ t\in C(I_{ab}).$$

We denote by K the cone of the positive functions from $C(I_{ab})$ and let us define a mapping

$$\|\cdot\|_*:C(I_{ab})\to K,$$

by $\|t\|_*=|t(x,y)|,\ (x,y)\in I_{ab},$

for all $t\in C(I_{ab})$.

It is obvious that $\|\cdot\|_*$ is a K-norm on $C(I_{ab})$, that is

(n_1) $\|t\|_*\ge\theta$, for each $t\in C(I_{ab})$, and $\|t\|_*=\theta$ if and only if $t=\theta$

(θ is the null function);

(n_2) $\|\lambda t\|_* = |\lambda| \cdot \|u\|_*$, for each $t \in C(I_{ab})$ and all $\lambda \in \mathbb{R}$;

(n_3) $\|t+s\|_* \le \|t\|_* + \|s\|_*$, for any $t, s \in C(I_{ab})$.

The space $C(I_{ab})$ endowed with the K-norm $\|\cdot\|_*$ is a K-normed space, denoted by X in the sequel.
By other hand, K is the positive cone of $C(I_{ab})$ endowed with the Chebischev's norm, which is monotone.

This means K is a normal cone and therefore $(X, \|\cdot\|_*)$ is a K-Banach space.The partial order induced by K on X is given by $s \le t$ if and only if $s(x,y) \le t(x,y)$, $\forall (x,y) \in I_{ab}$.

Let now define $\omega : K \to K$, given by

$$\omega(t(x,y)) = h(x,y,\langle|t^*|\rangle),\ (x,y) \in I_{ab}, \tag{3.14}$$

where $\langle|t^*|\rangle$ is a $nm+n+m$-dimensional vector corresponding to $\langle t\rangle$ constructed in the following way:

if $\langle t\rangle = \left(t, \dfrac{\partial t}{\partial x}, \dots, \dfrac{\partial^n t}{\partial x^n}, \dots, \dfrac{\partial^j t}{\partial y^j}, \dots, \dfrac{\partial^{n+j} t}{\partial x^n \partial y^j}, \dots, \dfrac{\partial^{m-1} t}{\partial y^{m-1}}, \dots\right)$

is denoted by $(t_{i,j})$, $i=0,1,\dots,n$, $j=0,1,2,\dots,m$, $i+j<n+m$,

where $t_{i,j}$ corresponds to $\dfrac{\partial^{i+j} t}{\partial x^i \partial y^j}$,

then, for $i=1,2,\dots,n$, $j=1,2,\dots,m$

$$t^{\omega}_{i,j} = \frac{1}{(i-1)!\,(j-1)!}\int_0^x\int_c^y (x-\xi)^{i-1}(y-\zeta)^{j-1}|t_{i,j}|\,d\zeta\, d\xi,$$

for $i=1,2,\dots,n$

$$t^{\omega}_{i,0} = \frac{1}{(i-1)!}\int_0^x (x-\xi)^{i-1}|t_{i,0}|\,d\xi$$

and for $j=1,2,\dots,m$

$$t^{*}_{0,j}=\frac{1}{(j-1)!}\int_0^y (y-\zeta)^{j-1}\,|t_{0,j}|\,d\zeta .$$

Let us assume that there exist a number $\alpha, 0\le\alpha<1$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} a_k$, such that,begining from a fixed rank the following inequality

$$\omega^{k+1}(t)\le\alpha\,\omega^{k}(t)+a_k,\quad \text{for every}\quad t\in K, \tag{3.15}$$

holds, where ω^k stands for the k^{th} iterate of ω.

The main result of our paper may be now stated in

THEOREM 3.1. *Let problem (3.1)+(3.3) have an approximate solution* $\hat{t}(x,y)$ *and let in* $S(\hat{t},r)\subseteq X$, *assumptions* $(a_1)-(d_1)$ *hold. If,in addition* $\omega: K\to K$ *given by (3.14) satisfies (3.15) and*

$$\|\hat{t}-g(x,y,\langle B_{n,m}\hat{t}\rangle)\|\le\delta_2<(1-\alpha)r,$$

then the following hold:

(i) there exists a unique solution $t(x,y)$ *of problem (3.1)+(3.3) in* $S(\hat{t},r)$ *and hence in view of*

$$u(x,y)=B_{n,m}t(x,y),$$

problem (3.1)+(3.3) has a unique solution;

(ii)the sequence of successive approximations defined by

$$u_0(x,y)=B_{n,m}\hat{t}(x,y);$$

$$t_{k+1}(x,y)=g(x,y,\langle u_k\rangle),$$

$$u_{k+1}=B_{n,m}t_{k+1},\quad k=0,1,2,\dots$$

converges to $u(x,y)$ *with*

$$\|u_k-u\|_{*}\le\sigma(\|u_k-u_k\|),$$

where $\sigma(t)$ *is the sum of the series*

$$\sum_{k=0}^{\infty}\omega^{k}(t). \tag{3.16}$$

Proof. It suffices to show that the operator

$$T_2 : S(\hat{t}, r) \to X$$

defined by (3.10) satisfies the conditions of Theorem 2.2.

To this end let $\bar{t}, t \in S(\hat{t}, r)$.

Then,

$$|(T_2\bar{t})_{(x,y)} - (T_2 t)_{(x,y)}| = |g(x, y, \langle B_{n,m}\bar{t}\rangle) - g(x, y, \langle B_{n,m}t\rangle)|$$

and by (c_1),

$$|(T_2\bar{t})_{(x,y)} - (T_2 t)_{(x,y)}| \le h(x, y, |B_{n,m}\bar{t} - B_{n,m}t|), \quad (x,y) \in I_{ab}.$$

In view of (3.6-1), for $i=1,2,\ldots,n$, $j=1,2,\ldots,m$, we have

$$|B_{i,j}\bar{t} - B_{i,j}t| \le \frac{1}{(i-1)!\,(j-1)!}\int_0^x\int_0^y (x-\xi)^{i-1}(y-\zeta)^{j-1}|\bar{t}(\xi,\zeta) - t(\xi,\zeta)|\,d\xi\,d\zeta$$

for $i=1,2,\ldots,n$, from (3.6-3) we have

$$|B_{i,0}\bar{t} - B_{i,0}t| \le \frac{1}{(i-1)!}\int_0^x (x-\xi)^{i-1}|\bar{t}(\xi,y) - t(\xi,y)|\,d\xi$$

and, finally, for $j=1,2,\ldots,m$ from (3.6-2)

$$|B_{0,j}\bar{t} - B_{0,j}t| \le \frac{1}{(j-1)!}\int_0^y (y-\zeta)^{j-1}|\bar{t}(x-\zeta) - t(x,\zeta)|\,d\zeta.$$

Therefore, in view of (3.14), we conclude that

$$\|T_2\bar{t} - T_2 t\|_r \le \omega(\|\bar{t} - \check{t}\|), \quad \text{for all } \bar{t}, t \in S,$$

which shows that T_2 is a ω-contraction.
Then, all conditions in Theorem 2.2 are satisfied.
The proof is complete.

Remarks. 1) If ω is given by

$$\omega(t(x,y)) = \sum_{i=1,j=1}^{n,m} L_{n-i,m-j}|t_{i,j}| +$$

$$+\sum_{i=1}^{n} L_{n-i,m}|t_{i,0}| + \sum_{j=1}^{m} L_{n,m-j}|t_{0,j}|, \quad \forall (x,y) \in I_{ab},$$

with $L_{p,q}$ satisfying

$$\alpha = \sum_{\substack{i=0,j=0\\ i+j\neq 0}}^{n,m} L_{n-i,m-j} \frac{a^i b^j}{i!j!} < 1,$$

then, condition (3.15) holds with $a_k = 0, k = 0,1,2,\ldots$ and from Theorem 3.1 we obtain Theorem 3.2 [15].

2)Our condition (c_1) is of Perron type and is more general than the corresponding one in [15].

As a matter of fact the assumptions on the functions $\omega_{i,j}$ involved in the generalized Lipschitz condition (c_1) in [15] are too strong. If suffices to claim $\omega_{i,j}$ are subadditive nondecreasing and continuous at zero (the continuity of φ is an immediate consequence of the previous assumptions).

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