Buletinul Ştiinţific al Universităţii din Baia Mare
Seria B, Matematică-Informatică, vol. XI(1995), 55-58

Dedicated to the Centenary of "Gazeta Matematică"

ON THE GENERALIZED RATIO TEST

Vasile BERINDE and Josif SANDOR

Abstract. A generalization of the well-known ratio test for numerical series was studied in the first author's papers [1] and [2]. The aim of the present paper is to extend the generalized ratio test itself.

Keywords: series of positive terms, ratio test, generalized ratio test, Mertens' theorem.
AMS Subject Classification: 40 A 05

1. INTRODUCTION

This paper is concerned with the study of the generalized ratio test, given by the following theorem (see [1] or [2])

THEOREM 1. *Let* $\sum_{n=1}^{\infty} U_n$ *be a series with positive terms.*

1) If there exist a convergent series of non-negative terms $\sum_{n=1}^{\infty} V_n$ *and a constant number* k *such that*

$$\frac{U_{n+1}}{U_n+V_n} \le k < 1, \text{ ,for } n \ge N \text{ (fixed)}, \qquad (*)$$

then the series $\sum_{n=1}^{\infty} U_n$ *is convergent.*

2) If there exists a decreasing sequence of positive numbers such that for $n \geq N$ (fixed), we have

$$i)\ U_n > V_n \quad \text{and} \quad ii)\ \frac{U_{n+1}+V_n-V_{n+1}}{U_n} \geq 1,$$

then the series $\sum_{n=1}^{\infty} U_n$ is divergent.

Remarks. If the " comparison series" $\sum_{n=1}^{\infty} V_n$ in Theorem 1 is the null series, that is $V_n=0, n\geq 1$, ,then we obtain the well - known ratio test or D'Alembert's test.

As shown by some examples in [1] and [2], the generalized ratio test (given by Theorem 1) is better than the classical ratio test, because there exist series for which Theorem 1 applys and the ratio test does not apply.

However, the relevance of the generalized ratio test is rather theoretical, because in practise it is very difficult to find, for a given series $\sum_{n=1}^{\infty} U_n$, the suitable comparison series $\sum_{n=1}^{\infty} V_n$.

The " convergence part " of Theorem 1 may be generalized as in the next section.

2. Main result

The main result of this paper is given by

THEOREM 2. *Let $\sum_{n=1}^{\infty} U_n$ be a series of positive terms. If there exist a convergent series $\sum_{n=1}^{\infty} V_n$ and a sequence $(W_n)_{n\geq 1}$ satisfying*

$$(i)\quad 0 < W_n < 1,\ \ n\geq 1; \qquad (ii)\quad W_{n+m} \leq W_n \cdot W_m,\ \ \forall\ n,m \geq 1,$$

such that

$$W_n \cdot U_{n+1} \leq W_{n+1}(U_n+V_n),\ \ n\geq 1. \qquad (1)$$

then the series $\sum_{n=1}^{\infty} U_n$ *is convergent.*

Proof. From (1) and (i) we obtain

$$\frac{U_{n+1}}{W_{n+1}} \le \frac{U_n}{W_n} + \frac{V_n}{W_n}, \quad n \ge 1 . \tag{2}$$

If we successively take in (2), n:=1,2,...,m
and then we add these m inequalities, we finally obtain

$$U_m \le \frac{U_1}{W_1} \cdot W_{m-1} + \sum_{i=1}^{m} \frac{W_{m+1}}{W_i} \cdot V_i . \tag{3}$$

But, from (ii) we have

$$W_i \cdot W_{m-i} \ge W_m$$

hence, from (3) we deduce

$$U_m \le U_1 \cdot W_m + W_1 \cdot \sum_{i=1}^{m} V_i \cdot W_{m-i} . \tag{4}$$

Now, having in view that applying again ii) it results

$$W_{m+1} \le W_1^m$$

and that $0 < W_1 < 1$, we obtain the convergence of $\sum_{n=1}^{\infty} W^n$.

Therefore applying the Merten's theorem we deduce that $\sum_{n=1}^{\infty} U_n$ is convergent. The proof is complete.

Remarks. 1) For (W_n), given by $W_n = k^n$, $0 \le k < 1$, from Theorem 2 we obtain the generalized ratio test.

2) By means of the generalized ratio test we obtained a characterization of the series of decreasing positive terms [3], given by

CORROLARY 1. *A series of decreasing positive terms* $\sum_{n=1}^{\infty} U_n$ *is convergent if and only if there exist a convergent series of non-negative terms such that (*) is satisfied.*

It is easy to see that the same property holds if we replace (*) by (1). We thus obtain

CORROLARY 2. *A series of decreasing positive term* $\sum_{n=1}^{\infty} U_n$ *is convergent if and only if there exist a convergent series of non - negative terms* $\sum_{n=1}^{\infty} V_n$ *and a sequence* (W_n), *satisfying (i) and (ii), from Theorem 2 such that (1) holds.*

Proof. The " sufficiency " follows from Theorem 2. In order to prove the necessity, we take, $V_n = a \cdot U_n$, $n \geq 1$ $(a > 0)$ and $W_n = 1/(1+a)^n$, $n \geq 1$.

REFERENCES

1. BERINDE, V., Une généralization de critère de D'Alembert pour les séries positives, *Bul.Şt.Univ.Baia Mare*, vol VII (1991), 21 - 26
2. BERINDE, V., A convolution type proof of the generalized ratio test, *Bul.Şt.Univ.Baia Mare*, vol VIII (1992), 35 - 40
3. BERINDE, V., Generalized contractions and applications (in Romanian), *Ph.D.Thesis, Univ."Babeş-Bolyai" Cluj-Napoca*, 1993

University of Baia Mare
Victoriei 76, 4800 Baia Mare
ROMANIA

Porţeni Nr.79
4160, jud. Harghita
ROMANIA