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GEOMETRIC CONSEQUENCES OF A MEAN VALUE THEOREM

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Abstract. Let $F:D\subset\mathbb{R}^n\to\mathbb{R}^m$, $F=(F_1,\ldots,F_m)$; $x^*\in\mathbb{R}^n$; $y^*\in\mathbb{R}^m$. In this note we apply Lagrange's mean value theorem to the real function

$$f:D\to\mathbb{R},\ f(x)=\langle x-x^*,x-x^*\rangle+\langle F(x)-y^*,F(x)-y^*\rangle$$

and present a geometric interpretation in case of m=1 or n=1.
($\langle\cdot,\cdot\rangle$ denotes the canonical scalar product in $\mathbb{R}^p$, $p\in\mathbb{N}^*$)

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$$f:D\to\mathbb{R},\ f(x)=\langle x-x^*,x-x^*\rangle+\langle F(x)-y^*,F(x)-y^*\rangle \tag{1}$$

and present a geometric interpretation in case of $m=1$ or $n=1$.

($\langle\cdot,\cdot\rangle$ denotes the canonical scalar product in $\mathbb{R}^p$, $p\in\mathbb{N}^*$).

A vector from $\mathbb{R}^p$ ($p\geq2$) will be considered in the form of a row-vector.

For $a,b\in\mathbb{R}^p$, $a\neq b$ we denote, as usual, by $]a,b[$ ($[a,b]$) the open (closed) segment with the endpoints a,b:

$$]a,b[=\{x\in\mathbb{R}^p : x=a+t(b-a),\ t\in]0,1[\}$$
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We denote by J_F the Jacobian matrix of F and by ∇f the gradient vector of f:

$$J_F=\begin{bmatrix}(F_1)'_{x_1}\ldots(F_1)'_{x_n}\\ \ldots\\ (F_m)'_{x_1}\ldots(F_m)'_{x_n}\end{bmatrix}$$

$$\nabla f=(f'_{x_1},\ldots,f'_{x_n})$$

Proposition. *Let $[a,b]\subset D\subseteq\mathbb{R}^n$, D an open set. If the function $F:D\to\mathbb{R}^m$ is continuous on $[a,b]$ and its Jacobian matrix J_F exists in every point of $]a,b[$, then for any $y^*\in\mathbb{R}^m$ there exists a $c\in]a,b[$ such that*

$$\langle(F(c)-y^*)\cdot J_F(c)+c-\frac{a+b}{2},b-a\rangle=\langle\frac{F(a)+F(b)}{2}-y^*,F(b)-F(a)\rangle \qquad (2)$$

Proof. Let $y^*\in\mathbb{R}^m$. We define the real function f by (1) with an arbitrary $x^*\in\mathbb{R}^n$. The hypotheses of the proposition imply that the function f is continuous on $[a,b]$ and its gradient vector ∇f exists in every point of $]a,b[$. By Lagrange's mean value theorem it follows that there exists a $c\in]a,b[$ (c deppending on y^*) for which

$$\langle\nabla f(c),b-a\rangle=f(b)-f(a).$$

Replacing here

$$\nabla f(x)=2(x-x^*)+2\cdot(F(x)-y^*)\cdot J_F(x) \quad \text{and}$$

$$f(b)-f(a)=\langle b-a,\ a+b-2x^*\rangle+\langle F(b)-F(a),\ F(a)+F(b)-2y^*\rangle$$

we get the relation (2).

Corollary 1. $(n=1, m=1)$

If the function $F:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is continuous on $[a,b]$ and derivable on $]a,b[$, then for any $y^*\in\mathbb{R}$ there exists a $c\in]a,b[$ such that

$$(F(c)-y^*)\cdot F'(c)+c=\left(\frac{F(a)+F(b)}{2}-y^*\right)\cdot\frac{F(b)-F(a)}{b-a}+\frac{a+b}{2} \qquad (3)$$

Geometric interpretation of Corollary 1. In the plane $\mathbb{R}^2=\{(x,y):x,y\in\mathbb{R}\}$ we consider the points $A(a,F(a)), B(b,F(b))$.

Relation (3) assures us, that the normal line in the point $(c,F(c))$ to the graph of F:

$$x+(y-F(c))\cdot F'(c)-c=0,$$

the midperpendicular line of the chord [AB]:

$$(b-a)(x-\frac{a+b}{2})+(F(b)-F(a))\cdot(y-\frac{F(a)+F(b)}{2})=0$$

and the line

$$y=y^*$$

are concurrent.

Corollary 1 asserts that under its hypotheses, in the plane $\mathbb{R}^2$, from every point of the midperpendicular of the chord $[AB]$ one can draw at least a normal line to the curve

$$]\widehat{AB}[:\ y=F(x),\ x\in]a,b[.$$

Remark. If in Corollary 1 $F(b)=\pm F(a)$ and $y^*=0$, we get Kolumbán's result from [2].

Corollary 2. $(n>1, m=1)$

Let $n>1$, $[a,b]\subset D\subseteq\mathbb{R}^n$, D an open set. If the function $F:D\to\mathbb{R}$ is continuous on $[a,b]$ and its gradient vector ∇F exists in every point of $]a,b[$, then for any $y^*\in\mathbb{R}$ there exists a $c\in]a,b[$ verifying

$$\langle (F(c)-y^*)\cdot\nabla F(c)+c-\frac{a+b}{2}, b-a\rangle=\left(\frac{F(a)+F(b)}{2}-y^*\right)\cdot(F(b)-F(a)) \quad (4)$$

Geometric interpretation of Corollary 2. We denote, as usual, $x=(x_1,\ldots,x_n)$, $a=(a_1,\ldots,a_n)$, $b=(b_1,\ldots,b_n)$, $c=(c_1,\ldots,c_n)$

$$A=(a_1,\ldots,a_n,F(a)),\ B=(b_1,\ldots,b_n,F(b)).$$

The graph of the function F is the hypersurface $\mathcal{H}$ of the space $\mathbb{R}^{n+1}=\{(x_1,\ldots,x_n,y): x_1,\ldots,x_n,y\in\mathbb{R}\}$ with the parametric equation

$$\mathcal{H}: \quad y=F(x_1,\ldots,x_n),\ (x_1,\ldots,x_n)\in D.$$

Assume that $F'_{x_1},\ldots,F'_{x_n}$ are continuous on]a,b[.

Relation (4) assures us that the normal line in the point $(c_1,\ldots,c_n,F(c))$ to the surface $\mathcal{H}$:

$$\begin{cases} x_1+(y-F(c))\cdot F'_{x_1}(c)-c_1=0 \\ \cdots \\ x_n+(y-F(c))\cdot F'_{x_n}(c)-c_n=0, \end{cases}$$

the midperpendicular hyperplane of the chord $[AB]$:

$$\langle b-a,\ x-\frac{a+b}{2}\rangle+(F(b)-F(a))\cdot\left(y-\frac{F(a)+F(b)}{2}\right)=0$$

and the hyperplane

$$y=y^*$$

are concurrent.

Corollary 2 asserts that under its hypotheses, in the space $\mathbb{R}^{n+1}$, for every $y^*\in\mathbb{R}$ there exists at least a point

$(c_1,\ldots,c_n,F(c))$ of the arc $]\overset{\frown}{AB}[: y=F(x_1,\ldots,x_n),(x_1,\ldots,x_n)\in]a,b[$ of the hypersurface $\mathcal{H}$: $y=F(x_1,\ldots,x_n),(x_1,\ldots,x_n)\in D$ in which the normal line to the hypersurface $\mathcal{H}$ intersects the midperpendicular hyperplane of the chord [AB] in a point of the hyperplane $y=y^0$.

Corollary 3. $(n=1, m>1)$

Let $m>1$ .If the function $F:[a,b]\subset\mathbb{R}\to\mathbb{R}^m$ is continuous on $[a,b]$ and derivable on $]a,b[$, then for any $y^*\in\mathbb{R}^m$ there exists a $c\in]a,b[$ verifying

$$(F(c)-y^*)\cdot J_F(c)+c=<\frac{F(a)+F(b)}{2}-y^*,\frac{F(b)-F(a)}{b-a}>+\frac{a+b}{2} \qquad (5)$$

Geometric interpretation of Corollary 3. Let us denote

$$y=(y_1,\ldots,y_m),\ y^*=(y_1^*,\ldots,y_m^*),$$

$$A=(a,F_1(a),\ldots,F_m(a)),\ B=(b,F_1(b),\ldots,F_m(b)).$$

The graph of the function F is the curve $[\widehat{AB}]$ of the space $\mathbb{R}^{m+1}=\{(x,y_1,\ldots,y_m):x,y_1,\ldots,y_m\in\mathbb{R}\}$ with the parametric description

$$[\widehat{AB}]:\ \begin{cases} x=x\\ y_1=F_1(x)\\ \ldots\\ y_m=F_m(x)\end{cases},\quad x\in[a,b].$$

Relation (5) assures us that the normal hyperplane to the curve $[\widehat{AB}]$ in the point $(c,F_1(c),\ldots,F_m(c))$:

$$x+(y-F(c))\cdot J_F(c)-c=0,$$

the midperpendicular hyperplane of the chord $[AB]$:

$$\left(x-\frac{a+b}{2}\right)+<\frac{F(b)-F(a)}{b-a},\ y-\frac{F(a)+F(b)}{2}>=0$$

and the line

$$\begin{cases} y_1=y_1^*\\ \ldots\\ y_m=y_m^*\end{cases}$$

are concurrent.

Corollary 3 asserts that under its hypotheses, in the space

$\mathbb{R}^{n+1}$, from every point of the midperpendicular hiperplane of the chord [AB] one can draw at least a normal hyperplane to the curve

$$]\widehat{AB}[: \begin{cases} x=x \\ y_1=F_1(x) \\ \dots \\ y_n=F_n(x) \end{cases}, \; x\in]a,b[.$$

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