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DISTRIBUTIONS IN THE CASE OF HOLOMORPHIC FUNCTIONS. AN IMPULSE FUNCTION (DIRAC`S DISTRIBUTION) FOR HOLOMORPHIC FUNCTIONS

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Abstract : In the paper a Dirac`s distribution for holomorphic functions is constructed ; it has the form:

$$(\delta(z-z_0), f(z)) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-z_0}dz = f(z_0), where \quad f \in H(D)$$

The Cauchy integral has the properties of a Dirac distribution; the filtering property is proved to hold and the derivative in distribution sense is calculated.

The derivative in distribution sense of the distribution generated by a function f satisfies the well-known relation : $f'(z) = \tilde{f}(z) + s_0(f)\ \delta\ (z - z_0)$

where $f'(z)$ is the derivative in distribution sense of the distribution generated by f, $\tilde{f}(z)$ is the distribution function-type of the function`s derivative and $s_0(f)$ is the function`s saltus on $\gamma \subset D$.

Definition 1: Let $f:C \to C$ be a function and denote by **supp** $f = \{z \in C | f(z) \neq 0\}$ the support of the function f(z) .Let K be the space of holomorphic function $f:C \to C$ having compact support and define uniform convergence in K as follows:

$$\varphi_n^{(k)}(z) \underset{n \quad K}{\overset{u.c.}{\to}} \varphi^{(k)}(z) \quad if \quad \exists \quad D \subset C \text{ such that}$$

supp $(\varphi_n(z)) \subset D$, **supp** $(\varphi(z)) \subset D, \forall k \in N$.

Definition 2 :Let S be the space of temperate functions , i.e. $\varphi \in S$ *if* $\varphi:C \to C$, $\varphi \in H(D), D \in C$ so that for $\|z\| \to \infty$ we have $\lim_{\|z\| \to \infty} zf(z) = 0$ (functions which vanish faster than any power of $\frac{1}{\|z\|}$).

Remark: In the case of holomorphic function Jordan's Lemma holds:

If $f(z) \in H(D)$ *and* $(\gamma_R): |z| = R, z = R = e^{i\theta}, \theta \in [\theta_1, \theta_2]$ *with* $\lim_{|z| \to \infty} zf(z) = 0$ (uniformly with respect to $\theta \in [\theta_1, \theta_2]$), then $\lim_{R \to \infty} \int_{\gamma_R} f(z)dz = 0$

Definition 3: We say that $f_n(z) \overset{s}{\to} f(z)$ if in any bounded domain D of C we have $f_n^{(k)}(z) \underset{D}{\overset{u.c.}{\longrightarrow}} f^{(K)}(z)$ and , moreover , $|z^k . D^\alpha f_i(z)| \le C_{k\alpha}$

(constants do not depend on function's index in the sequence).

Definition 4: Let ξ be the space of indefinite differentiable complex functions $f:C \to C$, having any support of function . Note that holomorphic functions belong to the space S.

In particular , for holomorphic functions - because zero are isolated - supp f(z) is a multiply connected domain , in fact the complex plane C except the points z_0 for which $f(z_0) = 0$.

Definition:5. Let $(E,C),(Y,C)$ be two vector spaces over the field Γ and $X \subset E$. A mapping $T:X \to Y$ is an operator with the domain X and value in Y. Denote $T(x) = (T,x) = y \in Y$.

T is a linear operator if
$T(\alpha_1 x_1+\alpha_2 x_2)=\alpha_1 T(x_1)+\alpha_2 T(x_2), for \quad \forall \alpha_i \in \Gamma, \forall x_i \in X$.

T is a functional if $Y=\Gamma$ (*the field* Γ), *i.e.* $T:X \to \Gamma$.

Definition 6. A distribution is a continuous linear functional defined on a fundamental space $\Phi(K,S,\xi)$, i.e. it maps any functional from Φ into a complex number , and this correspondence satisfies the following conditions:

(1) $(T,\alpha_1 f_1+\alpha_2 f_2)=\alpha_1(T,f_1)+\alpha_2(T,f_2)$

(2) $f_i \xrightarrow{\phi} f \Rightarrow \lim(T,f_i)=(T,f); \quad f_i,f \in \Phi$.

We shall denote by $T\in\phi'$ a distribution defined on ϕ.

The sequence of distribution T_i converges to the distribution T, $T\in\phi'$: $\lim_{i\to\infty} T_i = T \Leftrightarrow \lim_{i\to\infty}(T_i,f)=(T,f)$, i.e. the sequence of complex number (T_i,f) converges to the complex number (T,f).

The set ϕ' of distributions defined on ϕ with addition , scalar multiplication and a convergence structure is a vector space with convergence , named the space of distributions ϕ'.

We shall construct now a Dirac distribution (an impulse function).

Let us consider the class of distributions generated by functions which are locally integrable on C.

Example : $(T_f,\varphi)=\int_\gamma f(z)\varphi(z)dz \ ,\varphi\in K,\gamma\subset K$

Definition 7 :An impulse function $\delta(z):C$ is defined by :

$(\delta(z),f(z))=f(0), \quad f\in\phi, \ centered \quad in \ z=0$

or $(\delta(z-z_0),f(z))=f(z_0), \ centered \quad in \ z_0$

Extend in complex the Heaviside function (unit function) :

$$\theta(z)=\begin{cases}1 & ,z\in\gamma\\ 0 & ,z\notin\gamma\end{cases}$$

and define $(\theta(z),f(z))=\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$

The distribution : $(T_\theta, f(z)) = \int_\gamma \theta(z) f(z) dz = \int_\gamma f(z) dz, \quad where \ \ \gamma \subset \sup p\{f(z)\}$ will be called Heaviside distribution .

We shall define now the Dirac distribution of an holomorphic function:

$$f(z) \in H(D) \to C, \quad D \subset C$$

$$(\delta(z), f(z)) = \int_\gamma \delta(z) f(z) dz \stackrel{def}{=} \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z} dz = f(0)$$

or, centered in z_0:

$$(\delta(z - z_0), f(z)) \stackrel{def}{=} \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} dz = f(z_0) \ .$$

This distribution is a linear and continuous one and will have the same properties as Dirac's distribution in the real case.

Filtering property:

$$\psi\delta(z - z_0) = \psi(z_0)\delta(z - z_0), \quad for \quad \forall \psi \in H(D)$$

Indeed

$$(\psi(z)\delta(z - z_0), f(z) = \frac{1}{2\pi i} \int_\gamma \frac{\varphi(z) f(z)}{z - z_0} dz = \psi(z_0) f(z_0) = (\psi(z_0)\delta(z - z_0), f(z))$$

Derivative of the distribution :

$$\delta'(z - z_0), f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^2} dz = -f'(z_0) = -(\delta(z - z_0), f'(z)) \ .$$

In general:

$$(\delta^{(s)}, f(z)) = (-1)^s (\delta(z), f^{(s)}(z)) \quad (where \quad f'(z) = \frac{\partial f}{\partial z}) \ .$$

For the Heaviside distribution we have : $\frac{d\theta(z)}{dz} = \delta(z)$,

because $\theta(z) = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z - z_0} = \begin{cases} 1 & , z \in \gamma \qquad |z - z_0| = a \\ 0 & , \ z \ -outside \ \ the\, contour\ \gamma \end{cases}$

Remark : Let us remind a lemma concerning the following Dirichlet problem :

Find an holomorphic function $F(z) = U(z) + iV(z)$ inside $D \cup \gamma, \gamma = FrD$ except at $z_0 \in \gamma$ where $\text{Re}\{F(z)\}$ has a saltus $S(z_0) = U_2 - U_1$ (simple pole).

The solution of the problem will be : around the point z_0

$$F(z)=\frac{i}{\pi}(U_2-U_1)\ln(z-z_0)+\text{continuous function}$$

The derivative of this function is

$$F'(z)=-\frac{i}{\pi}(U_2-U_1)\frac{1}{z-z_0}+$$ continuous function or , in general , for z_n - simple poles and $S_n(z_n)=$ saltus of $\mathrm{Re}\{F(z)\}$, we have:

$$F(z)=G(z)+i\sum_{k=1}^{n}\frac{S_k(z_k)}{\pi}\ln(z_n-z),$$

G(z) being a continuous function .

Derivative in distribution sense in the neighbourhood of z_0 is a generalization of the usual derivative.

Proposition : (generalization of the real case)

Let $F(z)\in H(D)$ except at the point $z_0\in\gamma$, where F(z) has a pole with the saltus of $\mathrm{Re}\{F(z)\}$: $s_0(F(z))=U_2-U_1$

i.e. $F(z)=\begin{cases}U_2+iV & , on\ \gamma_2\\ U_1+iV & , on\ \gamma_1\end{cases}$

$s_0(F(z))=U_2-U_1$, then the following relation holds:

$$f'(z)=f'(z)+s_0(f)\delta(z-z_0)$$

where $f'(z)$ is the derivative (in distribution sense) of the distribution generated by f .

$f'(z)$ is the distribution function - type of the function`s derivative .

If

$$f(z)=\delta(z),\quad then\quad \delta'(z)=(\delta',f)\quad and\quad f'(z)=(\delta,f')$$

$$(f,\varphi)'_z=(f,\varphi')+s_0(f)(\delta(z-z_0),\varphi)$$

The point $z_0\in\gamma$, then $\int_\gamma\frac{dz}{z-z_0}=\pi i$ (semiresidue theorem).

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