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ON A MEAN VALUE THEOREM WITH DIVIDED DIFFERENCES

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Abstract. A mean value theorem for divided differences is presented. For the intermediary point, a similar property with the one proven by B. Jacobson [1] in the case of the mean value theorem for Riemann integrals is proven.

Mean value theorems relatively to real functions $f:[a,b]\to\mathbb{R}$ praise the existence of an intemediary point $c\in(a,b)$ for which Jacobson [1], in the case of the mean value theorem for the Riemann integral has proved the following property:

$$\lim_{b\to a}\frac{c-a}{b-a}=\frac{1}{2}$$

Popa [3] showed that this result spreads also in case of Lagrange's theorem and the second mean value theorem.

In the folowing we shall give a mean value theorem with divided differences which represents a generalization of the previous theorems and we shall prove a similar property for the intermediary point corresponding to this theorem.

In [2] we have proved the theorem:

Theorem 1. *If* $f_i:[a,b]\to\mathbb{R}$ *are* $n-1$ *times continuously differentiable on* $[a,b]$ *and* $f_i^{(n)}$ *also exist on* (a,b) *for every* $i=\overline{1,n+1}$, *then for any* $a\le x_0<x_1<\dots<x_n\le b$ *there exists at least an* $c\in(a,b)$ *such that:*

(1) $\sum_{i=0}^{n+1}(-1)^i f_i^{(n)}(c)D_i=0,$

where D_i means the determinant obtained through a removal of the i^{th} column from the $(n+1,n+2)$ type matrix:

$$(2)\quad \begin{Vmatrix} f_1(x_0) & \dots & f_{n+1}(x_0) & 1 \\ \dots & \dots & \dots & \\ f_1(x_n) & \dots & f_{n+1}(x_n) & 1 \end{Vmatrix}$$

The relation (1) can be written in n+2 order determinant form:

$$(3)\quad \begin{vmatrix} f_1^{(n)}(c) & \dots & f_{n+1}^{(n)}(c) & 0 \\ f_1(x_0) & \dots & f_{n+1}(x_0) & 1 \\ \dots & \dots & \dots & \dots \\ f_1(x_n) & \dots & f_{n+1}(x_n) & 1 \end{vmatrix}=0$$

Subtracting succesively the i^{th} line from the i+1 one, $i=\overline{2,n+1}$, remembering that $f_j(x_{i-1})-f_j(x_{i-2})=(x_{i-1}-x_{i-2})Df_j(x_{i-2})$ and developing the obtained determinant in accordance with the last column we get the following:

$$\prod_{i=0}^{n-1}(x_{i-1}-x_{i-2})\cdot\begin{vmatrix} f_1^{(n)}(c) & \dots & f_{n+1}^{(n)}(c) \\ Df_1(x_0) & \dots & Df_{n+1}(x_0) \\ \dots & \dots & \dots \\ Df_1(x_n) & \dots & Df_{n+1}(x_n) \end{vmatrix}=0$$

Repeating succesively this method for the lines i, where $i=\overline{2,n}$, then for $i=\overline{3,n}$, and so on we get the following:

$$V(x_0,x_1,\ldots,x_n)\cdot\begin{vmatrix} f_1^{(n)}(c) & \ldots & f_{n+1}^{(n)}(c)\\ Df_1(x_0) & \ldots & Df_{n+1}(x_0)\\ \ldots & \ldots & \ldots\\ D^nf_1(x_n) & \ldots & D^nf_{n+1}(x_n)\end{vmatrix}=0,$$

where $V(x_0,x_1,\ldots,x_n)$ means the Vandermonde determinant for $x_0,x_1,\ldots,x_n$ and $D^jf_i(x_k)$ the j^{th} order divided difference applied to the function f_i in the points $x_k,x_{k+1},\ldots,x_{k+j}$.

So, we have obtained the mean value theorem for the divided differences of the functions f_i, where $i=\overline{1,n+1}$:

Theorem 1': *If $f_i:[a,b]\to\mathbb{R}$ are $n-1$ times continuously differentiable on $[a,b]$ and there exist $f_i^{(n)}$ on (a,b) for every $i=\overline{1,n+1}$ then for any $a\le x_0<x_1<\ldots<x_n\le b$ there is at least an $c\in(a,b)$ such that:*

$$(4)\quad \begin{vmatrix} f_1^{(n)}(c) & \ldots & f_{n+1}^{(n)}(c)\\ Df_1(x_0) & \ldots & Df_{n+1}(x_0)\\ \ldots & \ldots & \ldots\\ D^nf_1(x_n) & \ldots & D^nf_{n+1}(x_n)\end{vmatrix}=0$$

Remark 1: *If $f:[a,b]\to\mathbb{R}$ is $n-1$ times continuously differentiable on $[a,b]$ and there exist $f^{(n)}$ on (a,b), then for any $a\le x_0<x_1<\ldots<x_n\le b$ there is at least an $c\in(a,b)$ for which:*

$$(5)\quad D^nf(x_0)=\frac{f^{(n)}(c)}{n!}.$$

This result represents the mean value theorem for the divided differences of order n and can be obtained by applying the previous theorem in the following case:

$$f_i(x)=x^i,\ i=\overline{1,n},$$

$f_{n+1} = f,$

remembering that:

$D^j f_i(x_0) = 0 \text{ if } j > i,$

$D^j f_i(x_0) = 1 \text{ if } j = i, \quad \text{where} \quad i = \overline{1,n}$

and:

$f_i^{(n)} = 0, \; i = \overline{1, n-1}$

$f_n^{(n)} = n!$

Taking into account the expression of the divided differences of order n for equidistant nodes, the Remark 1 leads us to the following result:

Remark 2: *If $f:[a,b]\to\mathbb{R}$ is $n-1$ times continuously differentiable on $[a,b]$ and $f^{(n)}$ also exists on (a,b) then for any $x_0 \in [a,b)$ and $h \in (0, \frac{b-x_0}{n}]$ there is at least an $c \in (a,b)$ for wich:*

$$(6) \quad \sum_{i=0}^{n} (-1)^{n-i} C_n^i f(x_0+ih) = h^n f^{(n)}(c)$$

Relative to the intermediary point c of this theorem we can prove the following property:

Theorem 2: *Let $f:[a,b]\to\mathbb{R}$ be an $n+1$ times continuously differentiable on $[a,b]$ function and $x_i = a+ih, \; i=\overline{1,n}$, with $h = \frac{x-a}{n}$. For every $x \in (a,b]$, there exists a point $c_x \in (a,x)$ such that:*

$$f^{(n)}(c) = n! \; D^n f(x_0)$$

If $f^{(n)}(a) \neq 0$ then the following takes place:

$$(7) \quad \lim_{x\to a} \frac{c_x - a}{x-a} = \frac{1}{2}.$$

Proof: Let $F, G: [0, \frac{b-a}{n}] \to \mathbb{R},$

$$F(h)=\sum_{i=0}^{n}(-1)^{n-i}C_n^i f(x_0+ih)-h^n f^{(n)}(x_0)$$

$$G(h)=h^{n+1}$$

In the proof we shall use the folowing relations, which are easy to demonstrate through complete induction after k, where $k\in\{0,\dots,n+1\}$, $n\in N^*$:

Let:

$$S_{k,n}=\sum_{i=0}^{n}(-1)^{n-i}i^k C_n^i$$

then we have:

a) $S_{k,n}=0,\ k=\overline{0,n-1}$

b) $S_{n,n}=n!$

c) $S_{n+1,n}=(n+1)!\cdot\frac{n}{2}$

With respect to the assumptions of the theorem we obtain that F and G are n+1 times continuously differentiable on $[0,\frac{b-a}{n}]$

Thus the following holds:

$$F^{(k)}(h)=\sum_{i=0}^{n}(-1)^{n-i}C_n^i i^k f^{(k)}(x_0+ih)-\frac{n!}{(n-k)!}h^{n-k}f^{(n)}(x_0),\ k=\overline{0,n}$$

Since F is n+1 times continuously differentiable on $[0,\frac{b-a}{n}]$ we obtain:

$$\lim_{h\to 0}F^{(k)}(h)=F^{(k)}(0),\ k=\overline{0,n+1}$$

So $\lim_{h\to 0}F^{(k)}(h)=f^{(k)}(a)\sum_{i=0}^{n}(-1)^{n-i}C_n^i i^k-\frac{n!}{(n-k)!}h^{n-k}$,

for any $k\in\{0,\dots,n+1\}$. Therefore:

$$\lim_{h\to 0}F^{(k)}(h)=f^{(k)}(a)\left[S_{k,n}-\frac{n!}{(n-k)!}h^{n-k}\right],\ k=\overline{0,n-1}$$

$$\lim_{h\to 0}F^{(n)}(h)=f^{(k)}(a)\,[S_{n,n}-n!],\ k=n$$

Due to the relations a) and b) we get:

$$\lim_{h\to 0} F^{(k)}(h)=0,\ k=\overline{0,n}$$

Similarly,

$$\lim_{h\to 0} F^{(n+1)}(h)=\sum_{i=0}^{n}(-1)^{n-i}C_n^i i^{n+1} f^{(n+1)}(x_0+ih)=$$

$$=f^{(n+1)}(x_0)S_{n+1,n}=f^{(n+1)}(a)\cdot\frac{(n+1)!n}{2}.$$

Clearly, $\lim_{h\to 0} G^{(k)}(h)=0,\ k=\overline{0,n}$

and $\lim_{h\to 0} G^{(n+1)}(h)=(n+1)!.$

Thus, through succesive application of the l'Hospital's rule we obtain:

$$(8)\quad \lim_{h\to 0}\frac{F(h)}{G(h)}=\lim_{h\to 0}\frac{F'(h)}{G'(h)}=\ldots=\lim_{h\to 0}\frac{F^{(n+1)}(h)}{G^{(n+1)}(h)}=\frac{n}{2}f^{(n+1)}(a)$$

But, proceeding from the definition of the functions F and G, we obtain:

$$\lim_{h\to 0}\frac{F(h)}{G(h)}=\lim_{h\to 0}\frac{\sum_{i=0}^{n}(-1)^{n-i}C_n^i f(x_0+ih)-h^n f^{(n)}(x_0)}{h^{n+1}}$$

Because of the second remark, for any $h\in[0,\frac{b-a}{n}]$, taking $x=a+nh$, it results the existence of a $c_x\in(a,x)$ such that:

$$\sum_{i=0}^{n}(-1)^{n-i}C_n^i f(x_0+ih)=h^n f^{(n)}(c_x).$$

So: $$\lim_{h\to 0}\frac{F(h)}{G(h)}=\lim_{h\to 0}\frac{h^n f^{(n)}(c_x)-h^n f^{(n)}(a)}{h^{n+1}}=$$

$$=\lim_{h\to 0}\frac{f^{(n)}(c_x)-f^{(n)}(a)}{c_x-a}\cdot\frac{c_x-a}{x-a}\cdot\frac{x-a}{h}.$$

But $x-a=nh$, so $h\to 0$ iff $x\to a$. Moreover, $c_x\in(a,x)$ so for $h\to 0$ we obtain $c_x\to a$. As f is (n+1)times continuously differentiable on [a,b],

$$\lim_{h\to 0}\frac{f^{(n)}(c_x)-f^{(n)}(a)}{c_x-a}=f^{(n+1)}(a).$$

So:

(9) $$\lim_{h\to 0}\frac{F(h)}{G(h)}=n\cdot f^{(n+1)}_{(a)}\cdot\lim_{x\to a}\frac{c_x-a}{x-a}.$$

In accordance with (8) and (9), because $f^{(n+1)}(a)\neq 0$ the statement (7) is proved.

This property of the intermediary point of this theorem can be extended for theorem 1' in the case of equidistant nodes as follows:

Theorem 3: *Let* $f_i:[a,b]\to\mathbb{R}$ *be* $\frac{n(n+2)}{2}+1$ *times continously differentiable on* $[a,b]$ *for every* $i=\overline{1,n+1}$ *and* $x_i=a+ih,\ i=\overline{0,n},\ h=\frac{x-a}{n}$. *Then there exists at least a* $c_x\in(a,x)$ *such that:*

(10) $$\begin{vmatrix} f_1^{(n)}(c_x) & \dots & f_{n+1}^{(n)}(c_x) \\ Df_1(x_0) & \dots & Df_{n+1}(x_0) \\ \dots & \dots & \dots \\ D^n f_1(x_n) & \dots & D^n f_{n+1}(x_n) \end{vmatrix}=0$$

It the following holds:

(11) $$\begin{vmatrix} f_1(a) & \dots & f_{n+1}(a) \\ \dots & \dots & \dots \\ f_1^{(n+1)}(a) & \dots & f_{n+1}^{(n+1)}(a) \end{vmatrix}\neq 0,$$

then:

(12) $$\lim_{x\to a}\frac{c_x-a}{x-a}=\frac{1}{2}$$

Proof: Let $F,G:[0,\frac{b-a}{n}]\to\mathbb{R}$.

$$(13)\quad \begin{cases} G(h) = h^{\frac{n(n+1)}{2}+1} \\ F(h) = 2!\ldots n!\,h^{\frac{n(n+1)}{2}+1} \begin{vmatrix} f_1^{(n)}(x_0) & \ldots & f_{n+1}^{(n)}(x_0) \\ Df_1(x_0) & \ldots & Df_{n+1}(x_0) \\ \ldots & \ldots & \ldots \\ D^n f_1(x_0) & \ldots & D^n f_{n+1}(x_0) \end{vmatrix} \end{cases}$$

We notice that an equivalent form for F can also be given:

$$(14)\quad F(h) = \begin{vmatrix} f_1^{(n)}(x_0) & \ldots & f_{n+1}^{(n)}(x_0) & 0 \\ f_1(x_0) & \ldots & f_{n+1}(x_0) & 1 \\ \ldots & \ldots & \ldots & \ldots \\ f_1(x_0+nh) & \ldots & f_{n+1}(x_0+nh) & 1 \end{vmatrix}$$

The proof of the equivalence of these forms can be made through the same method which brought us to the theorem 1'.

From the hypotesis of the theorem it follows that F and G are $\frac{n(n+1)}{2}+1$ times continuously differentiable on $[0,\frac{b-a}{n}]$.

For any $h\in(0,\frac{b-a}{n}]$, considering x = a + nh, there exists at least a $c_x\in(a,x)$ such that (10) holds. Then:

$$F(h) = F(h) - 0 = 2!\ldots n!\,h^{\frac{n(n+1)}{2}+1}\cdot \begin{vmatrix} f_1^{(n)} x_0 & \ldots & f_{n+1}^{(n)} x_0 \\ Df_1(x_0) & \ldots & Df_{n+1}(x_0) \\ \ldots & \ldots & \ldots \\ D^n f_1(x_0) & \ldots & D^n f_{n+1}(x_0) \end{vmatrix} -$$

$$-2!\ldots n!\,h^{\frac{n(n+1)}{2}+1}\cdot \begin{vmatrix} f_1^{(n)} c_x & \ldots & f_{n+1}(c_x) \\ Df_1(x_0) & \ldots & Df_{n+1}(x_0) \\ \ldots & \ldots & \ldots \\ D^n f_1(x_0) & \ldots & D^n f_{n+1}(x_0) \end{vmatrix} =$$

$$=2!\ldots n!h^{\frac{n(n+1)}{2}+1}\ \frac{x_0-c_x}{h}\begin{vmatrix}\frac{f_1^{(n)}(x_0)-f_1^{(n)}(c_x)}{x_0-c_x} & \ldots & \frac{f_{n+1}^{(n)}(x_0)-f_{n+1}^{(n)}(c_x)}{x_0-c_x}\\ Df_1(x_0) & \ldots & Df_{n+1}(x_0)\\ \ldots & \ldots & \ldots\\ D^nf_1(x_0) & \ldots & D^nf_{n+1}(x_0)\end{vmatrix}$$

Applying the mean value theorem for k^{th} order divided difference, $k=\overline{1,n}$ with the functions f_i, $i=\overline{1,n+1}$ we obtain the exiscence of $c_k^i\in(a,x_k)\subset(a,x)$ such that:

$f_i^{(k)}(c_k^i)=k!\cdot D^kf_i(x_0)$, $i=\overline{1,n+1}$, $k=\overline{1,n}$.

But $x_0-x=-nh$, so:

$$F(h)=-n\frac{x_0-c_x}{x_0-x}h^{\frac{n(n+1)}{2}+1}\cdot\begin{vmatrix}\frac{f_1^{(n)}(x_0)-f_1^{(n)}(c_x)}{x_0-c_x} & \ldots & \frac{f_{n+1}^{(n)}(x_0)-f_{n+1}^{(n)}(c_x)}{x_0-c_x}\\ f_1'(c_1^1) & \ldots & f_{n+1}'(c_1^{n+1})\\ \ldots & \ldots & \ldots\\ f_1^{(n)}(c_n^1) & \ldots & f_{n+1}^{(n)}(c_n^{n+1})\end{vmatrix}$$

So: $$\lim_{h\to 0}\frac{F(h)}{G(h)}=-\lim_{h\to 0}\frac{c_x-a}{c_x-a}\cdot\begin{vmatrix}\frac{f_1^{(n)}(x_0)-f_1^{(n)}(c_x)}{x_0-c_x} & \ldots & \frac{f_{n+1}^{(n)}(x_0)-f_{n+1}^{(n)}(c_x)}{x_0-c_x}\\ f_1'(c_1^1) & \ldots & f_{n+1}'(c_1^{n+1})\\ \ldots & \ldots & \ldots\\ f_1^{(n)}(c_n^1) & \ldots & f_{n+1}^{(n)}(c_n^{n+1})\end{vmatrix}$$

But $h\to 0$ iff $x\to a$, and since $c_k^i\in(a,x)$,if $h\to 0$ then also $c_k^i\to a$, $i=\overline{1,n+1}$, $k=\overline{1,n}$.

Moreover, f_i is $\frac{n(n+1)}{2}+1$ times continuously differentiable on [a,b], therefore $f_i^{(n)}$ is derivable in a,so:

$$\lim_{h\to 0}\frac{f_i^{(n)}(c_x)-f_i^{(n)}(a)}{c_x-a}=\lim_{c_x\to a}\frac{f_i^{(n)}(c_x)-f_i^{(n)}(a)}{c_x-a}=f_i^{(n+1)}(a),$$

because $a=x_0$ and $c_x \in (a,x)$.

Thus we have obtained:

$$(15)\quad \lim_{h\to 0}\frac{F(h)}{G(h)}=-n\begin{vmatrix} f_1^{(n+1)}(a) & \dots & f_{n+1}^{(n+1)}(a) \\ f_1(a) & \dots & f_{n+1}(a) \\ \dots & \dots & \dots \\ f_1^{(n)}(a) & \dots & f_{n+1}^{(n)}(a) \end{vmatrix}\cdot\lim_{h\to 0}\frac{c_x-a}{c_x-a}.$$

Clearly G is $\frac{n(n+1)}{2}+1$ times continuously differentiable therefore we get:

$$\lim_{h\to 0}G^{(k)}(h)=G^{(k)}(0)=0,\ k=\overline{0,\frac{n(n+1)}{2}}$$

$$(16)\quad \lim_{h\to 0}G^{\left(\frac{n(n+1)}{2}+1\right)}(h)=G^{\left(\frac{n(n+1)}{2}+1\right)}(0)=\left(\frac{n(n+1)}{2}+1\right)!$$

Referring to (14), in accordance with the rule of the derivation of the determinants, we can write:

$$F^{(k)}_{(h)}=\sum_{\substack{k_1+\dots+k_n=k\\ k,\in N}} M^k_{k_1,\dots,k_n}\begin{vmatrix} f_1^{(n)}(x_0) & \dots & f_{n+1}^{(n)}(x_0) & 0 \\ f_1(x_0) & \dots & f_{n+1}(x_0) & 1 \\ [f_1(x_0+h)]^{(k_1)} & \dots & [f_{n+1}(x_0+h)]^{(k_1)} & 1^{(k_1)} \\ \dots & \dots & \dots & \dots \\ [f_1(x_0+nh)]^{(k_n)} & \dots & [f_{n+1}(x_0+nh)]^{(k_n)} & 1^{(k_n)} \end{vmatrix}$$

where $M^k_{k_1,\dots,k_n}$ are some constants which do not depend on $f_1,\dots,f_{n+1}$, or x_0. $[f_i(x_0+jh)]^{(k_j)}$ means the k_j^{th} derivative of the function f_i in x_0+jh, with regard to h, while:

$1^{(k_j)}=0,\quad$ if $\quad k_j>0$

$1^{(k_j)}=1,\quad$ if $\quad k_j=0,\ j=\overline{1,n}$

But every function f_i is only $\frac{n(n+1)}{2}+1$ times continuously differentiable on [a,b], so every k_j cannot overtake $\frac{n(n+1)}{2}+1$. The maximal k which ensures us that for every $\{k_1,\ldots,k_n\}$ with $k_1+\ldots+k_n=k$ the condition above is fulfilled is $\frac{n(n+1)}{2}+1$

So we get:

$$F^{(k)}_{(h)}=\sum_{\substack{k_1+\ldots+k_n=k\\ k_j\in N}} M^k_{k_1,\ldots,k_n}\cdot\prod_{j=1}^{n} j^{k_j}\begin{vmatrix} f_1^{(n)}(x_0) & \ldots & f_{n+1}^{(n)}(x_0) & 0\\ f_1(x_0) & \ldots & f_{n+1}(x_0) & 1\\ f_1^{(k_1)}(x_0+h) & \ldots & f_{n+1}^{(k_1)}(x_0+h) & 1^{(k_1)}\\ \ldots & \ldots & \ldots & \ldots\\ f_1^{(k_n)}(x_0+nh) & \ldots & f_{n+1}^{(k_n)}(x_0+nh) & 1^{(k_n)} \end{vmatrix}$$

Because F is $\frac{n(n+1)}{2}+1$ times continuously differentiable we obtain:

$$\lim_{h\to 0}F^{(k)}_{(h)}=F^{(k)}_{(0)}=\sum_{\substack{k_1+\ldots+k_n=k\\ k_j\in N}} M^k_{k_1,\ldots,k_n}\cdot\prod_{j=1}^{n} j^{k_j}\cdot\begin{vmatrix} f_1^{(n)}x_0 & \ldots & f_{n+1}^{(n)}x_0 & 0\\ f_1(x_0) & \ldots & f_{n+1}(x_0) & 1\\ f_1^{(k_1)}(x_0) & \ldots & f_{n+1}^{(k_1)}(x_0) & 1^{(k_1)}\\ \ldots & \ldots & \ldots & \ldots\\ f_1^{(k_n)}(x_0) & \ldots & f_{n+1}^{(k_n)}(x_0) & 1^{(k_n)} \end{vmatrix}$$

Clearly the determinants obtained above vanish (having two identical lines) in the following situations:

a) $i,j\in\{1,\ldots,n\}$, $i\neq j$ such that $k_i=k_j$,or

b) $i\in\{1,\ldots,n\}$ such that $k_i=0$ or $k_i=n$.

We notice that for every $k\in\{0,\ldots,\frac{n(n+1)}{2}\}$, each of the determinants which occur above are either in the situation a)or b). Therefore in this case

(17) $$\lim_{h\to 0}F^{(k)}_{(h)}=0.$$

If $k=\frac{n(n+1)}{2}+1$ the singular situations in which the determinants are not doubtlessly zero are those for which $\{k_1,\ldots,k_n\}$ represents a permutation of $\{1,\ldots,n-1,n+1\}$.

In each of these situations, through interchanging the positions of the lines and replacing x_0 with a we bring the determinants to the form:

$$(18)\quad \begin{vmatrix} f_1^{(n+1)}(a) & \dots & f_{n+1}^{(n+1)}(a) & 0 \\ f_1(a) & \dots & f_{n+1}(a) & 1 \\ f_1'(a) & \dots & f_{n+1}'(a) & 0 \\ \dots & \dots & \dots & 0 \\ f_1^{(n)}(a) & \dots & f_{n+1}^{(n)}(a) & 0 \end{vmatrix}$$

Let $T_{k_1,\dots,k_n}$ be the number of the line interchangings through which the following determinant can be brought to the form (18):

$$\begin{vmatrix} f_1^{(n)}(x_0) & \dots & f_{n+1}^{(n)}(x_0) & 0 \\ f_1(x_0) & \dots & f_{n+1}^{(}x_0) & 1 \\ f_1^{(k_1)}(x_0) & \dots & f_{n+1}^{(k_1)}(x_0) & 1^{(k_1)} \\ \dots & \dots & \dots & \dots \\ f_1^{(k_n)}(x_0) & \dots & f_{n+1}^{(k_n)}(x_0) & 1^{(k_n)} \end{vmatrix}$$

So we can write:
(19)

$$\lim_{h\to 0} F_{(h)}^{(k)} = \begin{vmatrix} f_1^{(n+1)}(a) & \dots & f_{n+1}^{(n+1)}(a) \\ f_1'(a) & \dots & f_{n+1}'(a) \\ \dots & \dots & \dots \\ f_1^{(n)}(a) & \dots & f_{n+1}^{(n)}(a) \end{vmatrix} \sum_{(k_1,\dots,k_n)=(1,\dots,n-1,n+1)}^{k_{j\in N}} M_{k_1,\dots,k_n}^{k} \prod_{j=1}^{n} j^{k_j}\, T_{k_1,\dots,k_n}$$

Because of (16), applying $n+1$ times the l'Hospital rule we get:

$$(20)\quad \lim_{h\to 0}\frac{F(h)}{G(h)} = \lim_{h\to 0}\frac{F^{\frac{n(n+1)}{2}+1}(h)}{G^{\frac{n(n+1)}{2}+1}(h)} =$$

$$= \frac{1}{\left[\frac{n(n+1)}{2}+1\right]!}\cdot \begin{vmatrix} f_1^{(n+1)}(a) & \dots & f_{n+1}^{(n+1)}(a) \\ f_1'(a) & \dots & f_{n+1}'(a) \\ \dots & \dots & \dots \\ f_1^{(n)}(a) & \dots & f_{n+1}^{(n)}(a) \end{vmatrix} \cdot \sum_{(k_1,\dots,k_n)=(1,\dots,n-1,n+1)}^{k_{j\in N}} M_{k_1,\dots,k_n}^{k} \prod_{j=1}^{n} T_{k_1}$$

(15) and (20), owing to (11)are leading us at the following relation:

$$(21)\quad \lim_{h\to 0}\frac{c_x-a}{x-a}=\lim_{x\to a}\frac{c_x-a}{x-a}=$$

$$=-\frac{1}{n\left[\frac{n(n+1)}{2}+1\right]!}\cdot\sum_{(k_1,\ldots,k_n)\in\{1,\ldots,n-1,n+1\}} M^{k}_{k_1,\ldots,k_n}\prod_{j=1}^{n} j^{k_j}(-1)\,T_{k_1,\ldots,k_n}$$

Clearly, for any functions f_i $i=\overline{1,n+1}$, the values of $T_{k_1,\ldots,k_n}$ and $M^{k}_{k_1,\ldots,k_n}$ are constant (they do not depend on those functions).So the value of the limit (21) does not depend on the selection of the functions f_i $i=\overline{1,n+1}$, thus we can determin it more precisely through a particular choice of them.

Let $f_i:[a,b]\to\mathbb{R}$, $f_i=x^i$, $i=\overline{1,n+1}$.

Note that in this case all the assumptions of the theorem are satisfied and also the assumptions of the second theorem and of first remark with $f:[a,b]\to\mathbb{R}$, $f=x^{n+1}$.

But because $f^{(n)}(c_x)=(n+1)!\cdot c_x$,we notice that the intermediary point obtained in the first remark is unique for any $x\in(a,b]$.

Theorem 2, applied in the same conditions leads us to the relation (7).Applying theorem 3 we get the relation (21).

Because of the uniqueness of the intermediary point, the two limits which appear in both of these relations are equal.Thus we have demonstrated the expected relation,namely (12).

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