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A FINITE ELEMENT METHOD FOR FRICTIONAL CONTACT PROBLEMS

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ABSTRACT. The present paper is concerned with the analysis with finite element of a friction contact phenomena for two elastic bodies that come into contactwith, friction obeying the normal compliance law. Variatonal principles for a class of friction contact problems are also established and finite element models and numerical algorithms for analyzing of this problem are presented. A perturbed Lagrangian discrete formulation within the framework of F.E.M. is obtained, and in the 3D case is used a four-nodes contact finite element which consists in 3 masters and 1 slave, generalizing the two dimensional case considered by Ju and Taylor [3] and by Wriggers and Simo [8]

1. INTRODUCTION.

The nature of dynamic friction forces developed between bodies in contact is extremely complex and is affected by a long list of factors : the constitution of the interface, the time scales and frequency of contact, the response of the interface to normal forces, inertia and thermal effects, roughness contacting surfaces, history of loadings, wear and general failure of the interface materials, the presence or absence of lubricants, and so on. Thus, dynamic friction is not a single phenomenon but is a collection of many complex mechanical and chemical phenomena.

Interface model for dynamic friction is the caracterization of the response of the interface to normal forces. This mechanical response for most metal-on-metal interfaces is highly nonlinear.

Stick-slip motion may be a manifestation of dynamic instabilities inherent in the coupling of normal and tangential relative motions of contacting bodies.

Finite element methods, together with numerical schemes for solving associated systems of nonlineare ordinary differential equations, are capable of modeling stick-slip motion, dynamic sliding, friction damping and related phenomena in a significant range of practical problems.

The new models of friction and contact, in the last decade, are often based on friction laws which recognize the compliant microstructure of contact interface,and that were not only more physically realistic than classical theories, but which were also mathematically tractable.

The existence of a solution for quasistatic frictional contact problems with normal compliance law was proved by Anderson [5] using incremental formulations and, in presence of a time regularization, by Klarbring et al.[6] in a different manner. Rabier et al. [7] proved the existence and local (for sufficiently small friction coefficients) uniqueness of solutions for cases in which sliding contact occurs in a prescribed direction.

The present paper is a continuation of the analysis presented in [4], which consists in a numerical analysis of a quasistatic contact problem in linear elasticity with dry friction. The problem is intended to model the physical situation of two elastically deforming bodies that come into contact with friction obeying the normal compliance law.

First we give a classical and variational formulation of the continuos contact problem. After obtaining the continuos contact problem we derive the result and obtain an incremental formulation obtained by time discretization of the problem.

Then we consider a discrete variational formulation of the incremental problem using a perturbed Lagrangian functional.

Also, in the present paper is described a contact finite element in the three dimensional case, generalizing the two dimensional case considered by Ju and Taylor in [3] and by Wriggers and Simo in [8].

2. CLASSICAL AND VARIATIONAL FORMULATIONS OF PROBLEMS IN ELASTODYNAMICS.

We shall now formulate a class of initial-value problems in elastodynamics which include sliding friction effects. Let $\Omega^{\alpha} \subset R^{N}$, $\alpha = 1,2$, N=2,3, the domains occupied by two elastic bodies that come into contact with friction.

Let us denote by Γ^{α} the boudary of Ω^{α} and let $\Gamma^{\alpha}_{0}, \Gamma^{\alpha}_{1}, \Gamma^{\alpha}_{2}$ be open and disjoint parts of Γ^{α} so that $\Gamma^{\alpha} = \Gamma^{\alpha}_{0} \cup \Gamma^{\alpha}_{1} \cup \Gamma^{\alpha}_{2}$ with $\alpha = 1,2$.

Assume that the bodies Ω^{α} are subjected to volume forces of density $f^{\alpha} = (f_1^{\alpha},...,f_N^{\alpha})$ on Ω^{α}, to surface tractions of density $t^{\alpha} = (t_1^{\alpha},...,t_N^{\alpha})$ on Γ_1^{α} and are hold fixed on Γ_0^{α}. We shall usse the following notations for the normal and tangential components of the displacements and of the stress vector:

$$u_n^{\alpha} = u_i^{\alpha} n_i^1, u_{t_i}^{\alpha} = u_i^{\alpha} - u_n^{\alpha} n_i^1, \sigma_n^{\alpha} = \sigma_{ij}^{\alpha} n_i^{\alpha} n_j^1, \sigma_{t_i}^{\alpha} = \sigma_{ij}^{\alpha} n_j - \sigma_n^{\alpha} n_i^1,$$

where i,j = 1,...,N, $n^{\alpha} = (n_1^{\alpha},...,n_N^{\alpha})$ is the outward normal unit vector on Γ^{α} and the summation convention is used for i and j.

Find the field of displacements $u^{\alpha} = (u_1^{\alpha},...,u_N^{\alpha})$, velocities $\dot{u}^{\alpha} = (\dot{u}_1^{\alpha},...,\dot{u}_N^{\alpha})$ and accelerations $\ddot{u} = (\ddot{u}_1^{\alpha},...,\ddot{u}_N^{\alpha})$ for a time interval [0,T], defineds on Ω^{α} wich satisfy the following equations and conditions :

- the equilibrium equation

$$\sigma^{\alpha}_{ij,j}(u^{\alpha}) + f_i^{\alpha} = \rho \ddot{u}_i^{\alpha} \text{ in } \Omega^{\alpha} \times (0,T) \tag{1}$$

- the constitutive equation

$$\sigma_{ij}^{\alpha} = a^{\alpha}_{ijkh} \varepsilon_{kh}(u^{\alpha}) \quad in\ \Omega^{\alpha} \tag{2}$$

where $a^{\alpha}_{ijkh} = a^{\alpha}_{jikh} = a^{\alpha}_{khij}$ and $a^{\alpha}_{ijkh}\xi_{ij}\xi_{kh} \geq c|\xi|$, $\xi = (\xi_{ij})$, and

$\varepsilon_{kh}(u^{\alpha}) = \frac{1}{2}(\frac{\partial u_k}{\partial x_h} + \frac{\partial u_h}{\partial x_k})$, f_i = the components of body force per unit

volume, assumed to be sufficiently smooth functions of $x = (x_1,...,x_N)$;

ρ = mass density,

$\frac{\partial \rho}{\partial t} = 0,\ \rho \in L^{\infty}(\Omega),\ \rho \geq \rho_0 > 0;$

$\ddot{u}_i$ = particle acceleration $\equiv \partial u^2/\partial t^2$;

- the boundary conditions

$$u^{\alpha}_i = 0 \ \ on \quad \Gamma^{\alpha}_0 \times (o,T)$$

$$\sigma^{\alpha}_{ij}(u^{\alpha})n_j = t^{\alpha}_i \ \ on \ \Gamma^{\alpha}_1 \times (0,T) \tag{3}$$

- the initial conditions

$$u^{\alpha}(x,0) = u^{\alpha}_o, \ \ u^{\alpha}_1(x,0) = u'^{\alpha}_o \ \ in \ \Omega^{\alpha} \ \ at \ t = 0 \tag{4}$$

with u^{α}_o, u^{α}_1 given smoth functions of x;

- the normal normal interface response

$$\sigma_n(u^{\alpha}) = -c_n(u^1_n - u^2_n - g)^{m_n}_+ \ on \ \Gamma^{\alpha}_2 \times (0,T) \tag{5}$$

with c_n and m_n material parameters (see [2]),

$$\text{or } \sigma_n = \frac{c_1\,(1617646.152\,\sigma/m)^{c_2}}{5.589^{1+0.0711c_2}} \exp[-\frac{1+0.0711c_2}{(1.363\sigma)^2} d^2]$$

where ξ is the initial mean plan distance d=ξg, c_1 and c_2 are mechanical constants expressing the nonlinear distribution of the surface hardness, σ and m are statistical parameters of the surface profile, representing respectively the RMS surface roughness and the mean asperity slope.

- the friction and contact conditions:

$$u^1_n - u^2_n \leq g \ \Rightarrow \ \sigma_T(u^{\alpha}) = 0$$

$$|\sigma_T(u^{\alpha})| \leq c_T(u^1_n - u^2_n - g)^{m_T}_+$$

$$u^1_n - u^2_n > g \ \Rightarrow \ |\sigma_T(u^{\alpha})| < c_T(u^1_n - u^2_n - g)^{m_T}_+ \Rightarrow u'^1_T - u'^2_T = U^{\alpha}_T \tag{6}$$

$$|\sigma_T(u^\alpha)| = c_T(u_n^1 - u_n^2 - g)_+^{m_T} \Rightarrow \exists\lambda \geq 0,\ u_T^{'1} - u_T^{'2} - U_T^{'C} = -\lambda\sigma_T \ on\ \Gamma_2^\alpha$$

Where c_n, m_n, c_T, m_T are material constants depending on interface proprerties, $b_+ = \max(0, b)$, $u_T^{'\alpha}$ is the tangential velocity of material particles on Γ_2^α, U_T^C is the prescribed tangential velocity of the Γ_2^1 with which Γ_2^2 comes in contact and g is the initial gap between Γ_2^1 and Γ_2^2 measured along the outward normal direction to Γ_2^1.

The friction law (6) is a generalization of the Coulomg's friction law, which is recovered if $m_n = m_T$. In such a case, $\mu = c_T/c_n$ is the usual coefficient of friction. The law (6) allows for a dependence of the friction coefficient on normal contact pressure.

Following steps similar to those of Duvaut and Lions [1], the nonlinear elastodynamics problem can be shown to be formally equivalent to the following variational problem:

Problem P1. Find the function $u = [u^1, u^2] : [0,T] \to V$ s. t.

$$\begin{aligned} &< u''(t), v - u'(t) > + a(u(t), v - u'(t)) + < P(u(t)), v - u'(t) > + \\ &+ j(u(t), v) - j(u(t), u'(t)) \geq < f(t), v - u'(t) >, \quad \forall\ v \in V \end{aligned} \tag{7}$$

with the initial conditions:

$$u(x, 0) = u_0\ ,\ \ u'(x, 0) = u_1 \tag{8}$$

We have assumed here, for simplicity, that $\rho \equiv 1$. The following notations and definitions were also used:

$$V = \{v^\alpha \in [H^1(\Omega^\alpha)]^N;\ v^\alpha = 0\ a.e.\ on\ \Gamma_0^\alpha\} \tag{9}$$

the space of admissible displacements (velocities)

$$a : V x V \to R$$

$$a(u, v) = \sum \int_\Omega a_{ijkl}^\alpha\, \varepsilon_{ij}^\alpha(u)\, \varepsilon_{kl}^\alpha(v)\, dx^\alpha \tag{10}$$

the virtual work prodused by the action of the stress $\sigma_{ij}(u)$ on the strains (strain rates) ε_{ij}

$$P : V \to V' \; < P(u), v > = \int_{\Gamma_2^\alpha} c_n (u_n^1 - u_n^2 - g)_+^{m_n} v_n ds \tag{11}$$

- the virtual work produced by the normal contact pressure on the displacement (velocity) v :

$$j : VxV \to R, \; j(u,v) = \int_{\Gamma_2^\alpha} c_T (u_n^1 - u_n^2 - g)_+^{m_T} |v_T^1 - v_T^2 - U_T^\alpha| ds \tag{12}$$

- the virtual power produced by the frictional force on the velocity v.

f(t) ∈ V'

$$< f(t), v > = \sum_{\alpha=1,2} \int_{\Omega^\alpha} f_i^\alpha v_i^\alpha dx^\alpha + \sum_{\alpha=1,2} \int_{\Gamma_1^\alpha} t_i^\alpha \gamma (v_i^\alpha) ds^\alpha \tag{13}$$

Here < . , > denotes duality pairing on V'xV where V' is the topological dual of V ; γ is the trace operator mapping $(H^1(\Omega))^N$ onto $(H^{1/2}(\Omega))^N$ which may be decomposed into normal component $\gamma_n(v)$ and tangential components $\gamma_T(v)$. Forsimplycity of notation, the lather are denoted as v_n and v_T, respectively. We also observe that the boundary integrals on Γ_2^α are well defined for $1 \le m_n, m_T \le 3$ if N=3 and for $1 \le m_n, m_T$ if N=2, because, for $v_i \in [H^1(\Omega)]^N$, $\gamma(v) \in [L^q(\Gamma_2^\alpha)]^N$, with $1 \le q \le 4$ for N=3, and with $1 \le q$ for N=2. In the case N=2 $m_n \in [2, 3.33]$ (see [2]).

The first step is to appoximate Problem P1 by a family of regularized problems which lead instead of a variational inequality. We approximate the friction functional j: VxV → R which is nondifferentiable in the second argument (velocity) by a family of functionals j_ε convex and differentiable on the second argument :

$j_\varepsilon : V \times V \to R$

$$j_\varepsilon (u, v) = \int_{\Gamma_2^\alpha} c_T (u_n^1 - u_n^2 - g)_+^{m_T} \psi_\varepsilon (v_T^1 - v_T^2 - U_T^\alpha) ds \tag{14}$$

where the function $\psi_\varepsilon : (L^q(\Gamma_2^q))^N \to L^q(\Gamma_2^q)$, is an approximation of the function $|.|^{\cdot} : (L^q(\Gamma_2^q))^N \to L^q(\Gamma_2^q)$ and is defined for $\varepsilon > 0, \xi \in (L^q(\Gamma_2^q))^N$ and a.e. $x \in \Gamma_2^q$, according to

$$\psi_\varepsilon(\xi) = \begin{cases} \varepsilon\left|\frac{\xi}{\varepsilon}\right|^2\left(1 - \frac{1}{3}\left|\frac{\xi}{\varepsilon}\right|\right) & \text{if } |\xi(x)| \leq \varepsilon \\ \varepsilon\left(\left|\frac{\xi}{\varepsilon}\right| - \frac{1}{3}\right) & \text{if } |\xi(x)| > \varepsilon \end{cases} \tag{15}$$

We now define the regularized form of Problem P1 :

Problem P1ε : Find the function $u_\varepsilon = [u_\varepsilon^1, u_\varepsilon^2] : [0, T] \to V$ s.t.

$$<u''_\varepsilon(t), v> + a(u_\varepsilon(t), v) + <P(u_\varepsilon(t), v> + <j_\varepsilon(u_\varepsilon(t), u'_\varepsilon(t)), v> =$$

$$= <f(t), v>, \forall v \in V \tag{16}$$

with the initial conditions

$$u_\varepsilon(x, 0) = u_o, \; u'_\varepsilon(x, 0) = u_1 \tag{17}$$

We observe that now have a variational equation instead of a variational inequality. However, the friction condition on Γ_2^q are now of the form :

Regularized friction conditions :

$$\sigma_T(u) = -c_T(u_s^1 - u_s^2 - g)_+^{m_T} F_\varepsilon \tag{18}$$

$$F_\varepsilon = \begin{cases} \left(2 - \left|\frac{u'^r_T - U'^C_T}{\varepsilon}\right|\right)\frac{u'^r_T - U'^C_T}{\varepsilon} & \text{if } |u'^r_T - U'^r_T| \leq \varepsilon \\ \frac{u'^r_T - U'^C_T}{|u'^r_T - U'^C_T|} & \text{if } |u'^r_T - U'^C_T| > \varepsilon \end{cases}$$

where $u'^r_T = u'^1_T - u'^2_T$

We consider now its particularization for the case of a two-dimensional ($N=2$) domain Ω^{α} with a boundary Γ_2^{α} sufficiently smoth that we can define a unit vector i_T tangent to Γ_2^{2}. In this case each vector ξ tangent to Γ_2^{2} is determined by the real number ξ such that $\xi = \xi i_T$. The functions ψ_ε and $\varphi_\varepsilon \equiv \psi'_\varepsilon$ are then, essentially, real-valued functions of a real variable, defined by

$$\Psi_\varepsilon(\xi) = \begin{cases} \varepsilon\,|\frac{\xi}{\varepsilon}|^2(1-\frac{1}{3}|\frac{\xi}{\varepsilon}|) & \text{if } |\xi| \le \varepsilon \\ \varepsilon(|\frac{\xi}{\varepsilon}| - \frac{1}{3}) & \text{if } |\xi| > \varepsilon \end{cases} \tag{19}$$

$$\Phi_\varepsilon(\xi) = \begin{cases} (2-|\frac{\xi}{\varepsilon}|)\frac{\xi}{\varepsilon} & \text{if } |\xi| \le \varepsilon \\ sgn(\xi) & \text{if } |\xi| > \varepsilon \end{cases} \tag{20}$$

The choice of ε will be dictated only by the desired proximity of the solutions of Problems P1 and P1ε and the corresponding computational costs associated.

3. FINITE ELEMENT APPROXIMATIONS OF THE CONACT PROBLEM.

Using standard finite element procedures, approximate version of Problem P1ε can be constructed in finite-dimensional subspaces $V_h(\subset V \subset V')$. For a certain (h) the approximate displacements, velocites and accelerations at each time t are elements of V_h

$$v^h(t), \dot{v}^h(t), \ddot{v}^h(t) \in V_h$$

Within each element $\Omega_e^h (e = 1,\dots,E_h)$ the components of the displacements velocities and accelerations are expressed in the form

$$v_i^h(x,t) = \sum_{I=1}^{N_e} v_i^I(t) N_I(x), \quad \dot{v}_i^h(x,t) = \sum_{I=1}^{N_e} \dot{v}_i^I(t) N_I(x), \quad \ddot{v}_i^h(x,t) = \sum_{I=1}^{N_e} \ddot{v}_i^I(t) N_I(x)$$

where $j=1,2,\ldots,N$; N_e= number of nodes of the element, $v_j^{\delta}(t), \dot{v}_j^{\delta}(t), \ddot{v}_j^{\delta}(t)$ are the nodal values of the displacements, etc., at time t and N_J is the element shape function associated wich the nodal I.

The finite element version of Problem P1ε is then :

Problem $P_{1\varepsilon}^{\delta}$: Find the function $u_\varepsilon^{\delta}:[0,T]\rightarrow V_\delta$ s.t.

$$(\ddot{u}_\varepsilon^{\delta}(t), v^{\delta}) + a(u_\varepsilon^{\delta}(t), v^{\delta}) + \langle P(u_\varepsilon^{\delta}(t)), v^{\delta}\rangle + \\ + \langle j_\varepsilon(u_\varepsilon^{\delta}(t), \dot{u}_\varepsilon^{\delta}(t)), v^{\delta}\rangle = \langle f(t), v^{\delta}\rangle, \forall\, v^{\delta}\in V^{\delta} \tag{21}$$

with the initial conditions

If N_δ is the number of nodes of finite element mesh , then this problem is equivalent to the following:

Find the function $r:[0,T]\rightarrow R^{N_e N_\delta}$, s.t.

$$M\ddot{r}(t) + K\,r(t) - P(r(t)) + j(r(t),\dot{r}(t)) = F(t), \tag{23}$$

with the initial conditions

$$r(0) = p_o,\ \dot{r}(0) = p_1 \tag{24}$$

Here we have introduced the following matrix notations;

$r(t), \dot{r}(t), \ddot{r}(t)$: the column vectors of nodal displacements, velocities and accelerations, respectively;

M: standard mass matrix;

K: standard stiffness matrix;

F(t): consistent nodal force vector;

P(r(t)): vector of consistent nodal normal forces on Γ_2^2;

$j(r(t),\dot{r}(t))$: vector of consistent nodal friction forces on Γ_2^2;

$p_o(p_1)$: initial nodal displacement (velocity);

The components of the element vector P are of the form

$$^{(e)}P = -\int_{^{(e)}\Gamma_2^2} \sigma_n\, n_j\, N_I\, ds \tag{25}$$

and the components of the element vector j are of the form

$$ {}^{(e)}j = -\int_{{}^{(e)}\Gamma_2^3} \sigma_{T_j} N_j \, ds \tag{26} $$

In order to obtain the components of the elements vectors P and j is used a contact finite element with 3 nodes which consists in 2 masters and 1 slave, in 2D case (see [3]), and a contact finite element with 4 nodes which consists in 3 masters and 1 slave in 3D case (see [4]).

In all numerical applications we derived a perturbed Lagrangian formulation for the case of frictional stick and for the case of frictional slide. For the case of frictional stick the perturbed Lagrangian functional for bodies in contact has the following form, in static case:

$$ \Lambda(u, \Sigma_n, \Sigma_t, \Sigma_\tau) = \frac{1}{2} a(u,u) - f(u) + \Sigma_n^T G_n + \Sigma_t^T G_t + \Sigma_\tau^T G_\tau - $$
$$ - \frac{1}{2\omega_n} \Sigma_n^T \Sigma_n - \frac{1}{2\omega_t} \Sigma_t^T \Sigma_t - \frac{1}{2\omega_\tau} \Sigma_\tau^T \Sigma_\tau \tag{27} $$

where u is the vector of nodal displacement, Σ_n, Σ_t, Σ_τ are the vectors of normal and tangential nodal contact forces, respectively, G_n, G_t, G_τ are the vectors of normal and tangential nodal gaps and ω_n, ω_t, ω_τ are the normal and tangential penalty parameters respectively.

The Newton-Raphson method was applied to the discrete variational formulations that can be derived from these perturbed Lagrangian functionals.

The normal vector on defined plane by the nodes 1, 2 and 3 and respectively vectors, defined by directions of the node 1-2 and 1-3 will be:

$$ n = \frac{(x_2 - x_1)(x_3 - x_1)}{|(x_2 - x_1)(x_3 - x_1)|}, \quad t = \frac{x_2 - x_1}{|x_2 - x_1|}, \quad \tau = \frac{x_3 - x_1}{|x_3 - x_1|} \tag{28} $$

where $x_1 = X_1 + u_1$, $x_2 = X_2 + u_2$, $x_3 = X_3 + u_3$ signify the current positions of master nodes; X_1, X_2, X_3 are reference coordinates and u_1, u_2, u_3 are current nodal displacements of points 1, 2 and 3.

In addition, we define the current 'surfaces coordinates' as following:

$$ a_t = \frac{x_s - x_1}{|x_2 - x_1|} t, \quad a_\tau = \frac{x_s - x_1}{|x_3 - x_1|} \tau \tag{29} $$

in which $x_s = X_s + u_s$ denotes the current position of the slave node s. The normal and tangential gaps g_n, g_t, g_τ are defined as:

where j=1,2,...N; N_e= number of nodes of the element,
$v_j(t), \dot{v}_j(t), \ddot{v}_j(t)$ are the nodal values of the displacements, etc., at time t and N_j is the element shape function associated wich the nodal I.

The finite element version of Problem P1ε is then :

Problem $P1_\varepsilon^h$: Find the function $u_\varepsilon^h : [0,T] \to V_h$ s.t.

$$(\ddot{u}_\varepsilon^h(t), v^h) + a(u_\varepsilon^h(t), v^h) + < P(u_\varepsilon^h(t)), v^h > + \\ + < j_\varepsilon(u_\varepsilon^h(t), \dot{u}_\varepsilon^h(t)), v^h > = < f(t), v^h >, \forall \; v^h \in V^h \tag{21}$$

with the initial conditions

If N_h is the number of nodes of finite element mesh , then this problem is equivalent to the following :

Find the function r : [0,T] $\to R^{N_e N_h}$, s.t.

$$M\ddot{r}(t) + K\,r(t) - P(r(t)) + j(r(t), \dot{r}(t)) = F(t), \tag{23}$$

with the initial conditions

$$r(0) = p_o;\ \dot{r}(0) = p_1 \tag{24}$$

Here we have introduced the following matrix notations :

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where u is the vector of nodal displacement, Σ_n, Σ_t, Σ_τ are the vectors of normal and tangential nodal contact forces, respectively, G_n, G_t, G_τ are the vectors of normal and tangential nodal gaps and ω_n, ω_t, ω_τ are the normal and tangential penalty parameters respectively.

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The normal vector on defined plane by the nodes 1, 2 and 3 and respectively vectors, defined by directions of the node 1-2 and 1-3 will be:

$$n = \frac{(x_2 - x_1)(x_3 - x_1)}{|(x_2 - x_1)(x_3 - x_1)|}, \; t = \frac{x_2 - x_1}{|x_2 - x_1|}, \; \tau = \frac{x_3 - x_1}{|x_3 - x_1|} \tag{28}$$

where $x_1 = X_1 + u_1$, $x_2 = X_2 + u_2$, $x_3 = X_3 + u_3$ signify the current positions of master nodes; X_1, X_2, X_3 are reference coordinates and u_1, u_2, u_3 are current nodal displacements of points 1, 2 and 3.

In addition, we define the current 'surfaces coordinates' as following:

$$a_t = \frac{x_s - x_1}{|x_2 - x_1|} t, \quad a_\tau = \frac{x_s - x_1}{|x_3 - x_1|} \tau \tag{29}$$

in which $x_s = X_s + u_s$ denotes the current position of the slave node s. The normal and tangential gaps g_n, g_t, g_τ are defined as:

$$g_n = (x_s - x_1)n\,,\; g_t = (a_t - a_t^o)\,|x_2 - x_1|\,,\; g_\tau = (a_\tau - a_\tau^o)\,|x_3 - x_1| \quad (30)$$

where a_t^o and a_τ^o are the old surface coordinates at the last time step known.

Note that the gap g depends on the slave node s as well as on the master nodes 1, 2 and 3. Thus, the variation of the gap is obtained according to

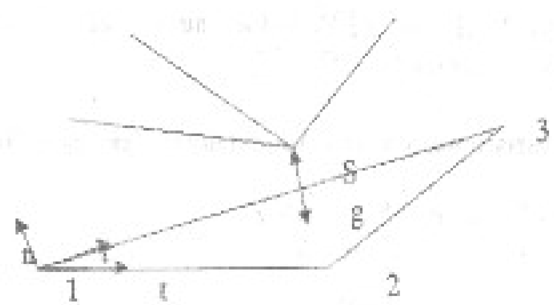

Fig. 1. Geometry of a three-dimensional contact element

$$g = \frac{d}{d\alpha}\, g\,(x_s + \alpha\,\eta_s\,, x_1 + \alpha\,\eta_1\,, x_2 + \alpha\,\eta_2, x_3 + \alpha\,\eta_3) \qquad (31)$$

where

$$\eta(\eta_1, \eta_2, \eta_3, \eta_s) \equiv \delta u(\delta u_1\,, \delta u_2, \delta u_3, \delta u_s) \qquad (32)$$

With respect to finite element implementations, explicit matrix expressions for the Lagrangian multiplier formulation and the penalty formulation are derived as follows.

The discrete variational equations associated with (27) take the form:

$$\delta_u \Pi(u) + \Sigma_n^T \delta_u G_n + \Sigma_t^T \delta_u G_t + \Sigma_\tau^T \delta_u G_\tau = 0 \qquad (33)$$

$$\delta \Sigma_n^T \left(-\frac{1}{\omega_n} \Sigma_n + G_n\right) = 0 \qquad (34)$$

$$\delta\Sigma_t^T\left(-\frac{1}{\omega_t}\Sigma_t+G_t\right)=0 \tag{35}$$

$$\delta\Sigma_\tau^T\left(-\frac{1}{\omega_\tau}\Sigma_\tau+G_\tau\right)=0 \tag{36}$$

where $\Pi(u)=\frac{1}{2}a(u,u)-f(u)$ is the total potential energy of the bodies in contact, $\delta_u G_n=(\delta_u g_n^1,\delta_u g_n^2,\ldots,\delta_u g_n^s)^T$, $\delta_u G_t=(\delta_u g_t^1,\delta_u g_t^2,\ldots,\delta_u g_t^s)^T$,

$\delta_u G_\tau=(\delta_u g_\tau^1,\delta_u g_\tau^2,\ldots,\delta_u g_\tau^s)^T$, S=total number of slave nodes in contact s=1,2,...,S, analogous for $\delta\Sigma_n,\delta\Sigma_t,\delta\Sigma_\tau$.

The variational of a typical nodal normal gap $g_n\in G_n$ take the form:

$$\delta g_n=\sum_{j=1}^{3}\frac{\partial g_n}{\partial u_s^j}\eta_s^j+\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial g_n}{\partial u_i^j}\eta_i^j$$

with the notation (32) and $c_n=\left(\frac{\partial g_n}{\partial u_s^1},\frac{\partial g_n}{\partial u_s^2},\frac{\partial g_n}{\partial u_s^3},\frac{\partial g_n}{\partial u_1^1},\ldots,\frac{\partial g_n}{\partial u_3^3}\right)$, $\eta=(\eta_s^1,\eta_s^2,\eta_s^3,\eta_1^1,\ldots,\eta_3^3)$, we obtain:

$$\delta g_n=\eta^T c_n$$

Similarly, the variation of a typical nodal tangential gap $g_t\in G_t$, $g_\tau\in G_\tau$ can be obtained according to

$$\delta g_t=\eta^T c_t,\ \delta g_\tau=\eta^T c_\tau$$

Moreover, the residual vector R_B and the tangent stiffness K_B associated with the total potential energy of the contacting bodies simply read, result

$$\delta\Pi(u)=\eta^T R_B \text{ and } \delta R_B=\eta^T K_B$$

With, the convention: $(u^1,\ldots,u^{12})=(u_s^1,u_s^2,u_s^3,u_1^1,\ldots,u_3^3)$ Eq. (33) become:

$$\eta^T\left[R_B+\sum_{s=1}^{S}(\sigma_n^s c_n^s+\sigma_t^s c_t^s+\sigma_\tau^s c_\tau^s)\right]=0 \tag{37}$$

and analogous for Eq.(34)-(35) where
$\sigma_n\in\Sigma_n,\sigma_t\in\Sigma_t,\sigma_\tau\in\Sigma_\tau$.

To apply the Newton's iteration scheme, consistent linearization of Eq.(37) and those corresponding Eq.(35)-(36), at $(u,\Sigma_n,\Sigma_t,\Sigma_\tau)$ is performed and leads to

$$[\eta^T, \delta\Sigma_n^T, \delta\Sigma_t^T, \delta\Sigma_\tau^T]\left\{\begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2^T & B_2 & O & O \\ A_3^T & O & C_3 & O \\ A_4^T & O & O & D_4 \end{bmatrix}\begin{Bmatrix} \Delta u \\ \Delta\Sigma_n \\ \Delta\Sigma_t \\ \Delta\Sigma_\tau \end{Bmatrix} = -\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix}\right\}$$

where $A_1 = K_B + \sum_{s=1}^{S}(k_n^s + k_t^s + k_\tau^s)$, $A_2 = \sum_{s=1}^{S} c_n^s$, $A_3 = \sum_{s=1}^{S} c_t^s$, $A_4 = \sum_{s=1}^{S} c_\tau^s$,

$$B_2 = -\frac{1}{\omega_n} I \quad , \quad C_3 = -\frac{1}{\omega_t} I \quad , \quad D_4 = -\frac{1}{\omega_\tau} I \quad ,$$

$$R_1 = R_B + \sum_{s=1}^{S}(\sigma_n^s c_n^s + \sigma_t^s c_t^s + \sigma_\tau^s c_\tau^s)$$

$$R_2 = -\frac{1}{\omega_n}\Sigma_n + G_n \quad , \quad R_3 = -\frac{1}{\omega_t}\Sigma_t + G_t \quad , \quad R_4 = -\frac{1}{\omega_\tau}\Sigma_\tau + G_\tau \quad ,$$

$$(k_n^s)_{ji} = \frac{\partial c_n^{si}}{\partial u_j} = \frac{\partial^2 g_n^s}{\partial u_i\,\partial u_j} \;,\; (k_t^s)_{ji} = \frac{\partial c_t^{si}}{\partial u_j} = \frac{\partial^2 g_t^s}{\partial u_i\,\partial u_j} \;,\; (k_\tau^s)_{ji} = \frac{\partial c_\tau^{si}}{\partial u_j} = \frac{\partial^2 g_\tau^s}{\partial u_i\,\partial u_j}$$

Finally after the discrete formulation within the framework FEM, a standard assembly procedure can be used to add the contact contributions of each contact element to the global tangent stiffness and residual matrix and thus we obtain:

$$K\,U = R \tag{38}$$

where $K = K_B + \sum_{s=1}^{S} K_C^s$, $R = -(R_B + \sum_{s=1}^{S} R_C^s)$, K_B, R_B are mechanical global tangent stiffness matrix and residual vector, K_C^s, R_C^s are mechanical contact contributions of contact nod s, $U = (\Delta u, \Delta\Sigma_n, \Delta\Sigma_t, \Delta\Sigma_\tau)^T$, S is the total number of the slave nodes. And for $\omega_n = \omega_t = \omega_\tau = \omega$ and $\sigma_n = \omega g_n$, $\sigma_t = \omega g_t$, $\sigma_\tau = \omega g_\tau$ result

$$K_C = \sum_{s=1}^{S}\omega(g_n^s k_n^s + g_t^s k_t^s + g_n^s k_\tau^s + c_n^{sT} c_n^s + c_t^{sT} c_t^s + c_\tau^{sT} c_\tau^s) \tag{39}$$

$$R_C = \sum_{s=1}^{S}\omega(g_n^{sT} c_n^s + g_t^{sT} c_t^s + g_\tau^{sT} c_\tau^s) \tag{40}$$

For the case of frictional slide the relation $|\Sigma_{tan}| = \mu|\Sigma_n|$, where μ is the coefficient of friction and Σ_{tan} is the result force of the Σ_t and Σ_τ, forces in the tangent plane of the contact surface.

Note with β the angle between the sides $\overline{x_2 - x_1}$ and $\overline{x_3 - x_1}$; we obtain $\cos\beta = t\tau$ and $|\lambda_{tan}| = \mu\sqrt{g_t^2 + g_\tau^2 + 2\varepsilon\,|g_t|\,|g_\tau|\cos\beta}$ where $\varepsilon = \mathrm{sgn}(g_t g_\tau)$. As a direct consequence of Coulomb's friction law, it results $\mu\omega|g_n| = \omega r$, where $r = \sqrt{g_t^2 + g_\tau^2 + 2\varepsilon\,|g_t|\,|g_\tau|\cos\beta}$ therefore

$$\lambda_t = \lambda_{tan}\frac{g_t}{r}\omega g_n = \quad = -\mu\,\mathrm{sgn}(g_t)\frac{g_t}{r}\omega g_n = -\mu\frac{|g_t|}{r}\omega g_n,$$

$\lambda_\tau = -\mu\dfrac{|g_\tau|}{r}\omega g_n$. If we note $d_t = \dfrac{|g_t|}{r}$, $d_\tau = \dfrac{|g_\tau|}{r}$, $b_t = \dfrac{\partial d_t}{\partial u}$, $b_\tau = \dfrac{\partial d_\tau}{\partial u}$, from linearized kinematics (i.e., by neglecting nonlinear terms k_t and k_τ), we obtain: $K_C = \sum_{s=1}^{S}(SL_1^s + SL_2^s)$, with

$$SL_1^s = \omega(g_n^s k_n^s - \mu g_n d_t^s k_t^s - \mu g_n^s d_\tau^s k_\tau^s + c_n^{sT} c_n^s - \mu d_t^s c_n^{sT} c_t^s),$$

$$SL_2^s = \omega(-\mu d_t^s c_n^{sT} c_\tau^s - \mu g_n b_t^{sT} c_t^s - \mu g_n b_\tau^{sT} c_\tau^s), \text{ and}$$

$$R_C = \sum_{s=1}^{S}\omega(\mu g_n d_t^{sT} c_t^s + \mu g_n d_\tau^{sT} c_\tau^s - g_n^{sT} c_n^s)$$

4. ALGORITMS FOR NONLINEAR DYNAMICAL SYSEMS.

The algoritms that we shall use for solving the discrete dynamical system involve variants of standard schems in use in nonlinear structural dynamics calculations: the Newmark-type algoritm or the central-difference scheme.

Using the Newton-Raphson method to solve the variational equation obtained at time t_s introducing, into the variational equation $(P1_s^s)$, the relations which defines the Newmark-type algoritm or the central-difference scheme is obtained the following system of algebric linear equations to be solved at each iteration.

Remark. The discontinuity of the Coulomb's friction law at zero sliding velocity is a major source of computational dificulties in friction problems. Even though, in the algoritms described in this and the previous sections, a regularized form of that law is used, those difficulties cannot be completely avoided. The situation which may arise when using the methods described herewith a constant time step is the following: in unloading situations (passage from sliding to adhesion) the Newton-Raphson iterative techniques may fail to converge if ε is very small and the step too large. For small values of ε the radius of converge of the iterative scheme used is very small due to the steep changes in Φ_{ε} on the interval $[-\varepsilon, \varepsilon]$.

The critical situations arise in transitions from sliding to adhesion because it is then that the most important changes in the solution occur. One simple remedy for these difficulties is to decrease the time step until two succesive solutions are not too far apart.

We give numerical exemples in [4] and [11], the numerical solution is in good agreement with the Raous [10]. The computations have been carried out within the environment of the Finte Element Analysis Program (FEAP), see Zienkiewicz[12], using the contact finite element in 3D, presented in this paper.

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