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METRIC RELATION BETWEEN THE MEDIANS AND THE BISECTRIX OF A TRIANGLE

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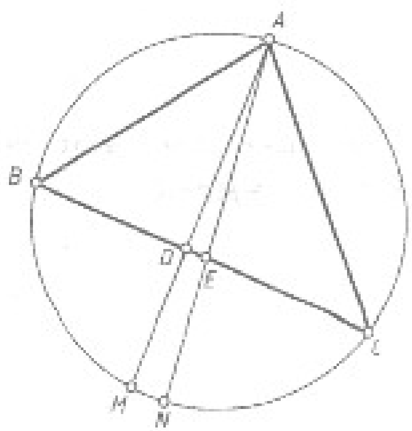
Abstract. This paper is meant to let you know about some relations between the medians and the bisectrix of a triangle. In order to do that, one resorts to the common notations used for any triangle ABC, that is l_a stands for the length of the bisector of the angle $\hat{A}$, and m_a stands for the median corresponding to the side l_b, l_c, m_b, m_c are analogous notations.

Theorem 1: *In any triangle ABC, there is the relation:*

$$\vec{m}_a \cdot \vec{l}_a = p(p-a)$$

Proof. In the triangle ABC one draws the bisector $AD = l_a$, the median $AE = m_a$ and then simedian $AF = s_a$. Applying Stewart's relation to the triangle ABC with the Cevian AF, one gets



$$c^2FC+b^2BF-s_a^2a=aBF\cdot FC \qquad (1)$$

On the other hand , from the relation $\frac{FC}{FB}=\frac{b^2}{c^2}$ one obtain the relations $FC=\frac{ab^2}{b^2+c^2}$ and $BF=\frac{ac^2}{b^2+c^2}$.

From the last two relations as well as from the relation (1) one infers the relation $s_a^2=\frac{2b^2c^2}{b^2+c^2}-\frac{a^2b^2c^2}{(b^2+c^2)^2}$ from which one may conclude that $s_a^2=\frac{2b^2c^2}{(b^2+c^2)^2}\cdot\frac{2(b^2+c^2)-a^2}{4}$ (2)

From the relation (2) , taking into account the fact that $m_a^2=\frac{2(b^2+c^2)-a^2}{4}$ one infers the relation $s_a=\frac{2bc}{b^2+c^2}\cdot m_a$.

From the last relations results that $\frac{m_a}{s_a}=\frac{b^2+c^2}{2bc}$ from which one infers the relation $\frac{m_a+s_a}{s_a}=\frac{(b+c)^2}{2bc}$ (3)

As AD is a bisector in the triangle ABC results that $i_a=\frac{2bc\cos\frac{A}{2}}{b+c}$ from which one deduces that $i_a^2=\frac{4b^2c^2\cos^2\frac{A}{2}}{(b+c)^2}$.

From the last relation and the relatrion $\cos^2\frac{A}{2}=\frac{p(p-a)}{bc}$ results that $\frac{(b+c)^2}{2bc}=\frac{2p(p-a)}{i_a^2}$ (4)

On the orther hand AD is the bisector in the triangle EAF , too , and noting $\alpha=E\hat{A}D$ one gets $i_a=\frac{2m_as_a\cos\alpha}{m_a+s_a}$. From the last relation one infers that:

$$\frac{m_a+s_a}{s_a}=\frac{2m_a\cos\alpha}{i_a} \qquad (5)$$

From the relations (3), (4), (5), one infers the relation $m_ai_a\cos\alpha=p(p-a)$ from which results that : $\vec{m_a}\,\vec{i_a}=p(p-a)$

Theorem 2. *In any triangle ABC one gets the relation:*

$$\vec{m}_a\vec{i}_a+\vec{m}_b\vec{i}_b+\vec{m}_c\vec{i}_c=p^2$$

Proof. According to theorem 1 , one has got the relations

$$\vec{m}_a\vec{i}_a=p(p-a),\quad \vec{m}_b\vec{i}_b=p(p-b),\quad and\quad \vec{m}_c\vec{i}_c=p(p-c)$$

Adding these relations , that is adding each member one by one , by turn , one gets the relation

$$\vec{m}_a\vec{i}_a+\vec{m}_b\vec{i}_b+\vec{m}_c\vec{i}_c=p^2$$

Corollary / *Problem 0:139 in Gazeta Matematica , no.5/1980 , author L.Panaitopol /. In any triangle ABC one has got the inequality:*

$$m_a i_a+m_b i_b+m_c i_c\geq p^2$$

Proof. From the inequalities :

$$m_a i_a\geq\vec{m}_a\vec{i}_a,\quad m_b i_b\geq\vec{m}_b\vec{i}_b,\quad and\quad m_c i_c\geq\vec{m}_c\vec{i}_c$$

one infers the inequality

$$m_a i_a+m_b i_b+m_c i_c\geq\vec{m}_a\vec{i}_a+\vec{m}_b\vec{i}_b+\vec{m}_c\vec{i}_c$$

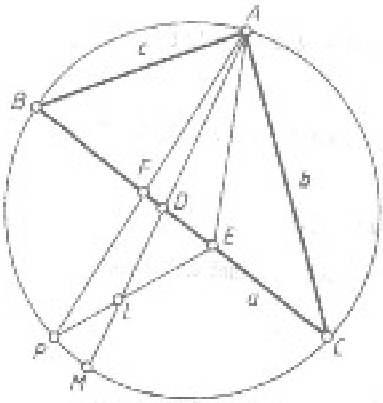
From the last inequality and from the theorem 2 it results

$$m_a i_a+m_b i_b+m_c i_c\geq p^2$$

Theorem 3. *If AD is the bisector and AE is the median of any triangle ABC and if M and N are the points in which the lines AD and AE intersect for the second time the circle circumscribed to the triangle ABC , then the projection of the segment (DM) on the line AN is congruous to (EN).*

Proof. From theorem 1 results that $m_a i_a\cos\alpha=p(p-a)$ from which one deduces the relation :

$$\frac{m_a i_a\cos\alpha}{bc}=\frac{p(p-a)}{bc}\qquad(1)$$



From the relation (1) and from relation $\frac{p(p-a)}{bc}=\cos^2\frac{A}{2}$ results that:

$$\frac{m_a i_a \cos\alpha}{bc}=\cos^2\frac{A}{2} \qquad (2)$$

From the relation (2) taking into account the fact that $bc=i_a AM$ on infers that $\frac{m_a \cos\alpha}{AM}=\cos^2\frac{A}{2}$ (3)

On the other hand , as triangle ECM is rectangular and $m\left(E\hat{C}M\right)=m\left(\hat{A}\right)/2$ results that $\cos\frac{A}{2}=\frac{a}{2CM}$ (4)

From the relations (3) and (4) one deduces the relation

$$\frac{m_a \cos\alpha}{AM}=\frac{a^2}{4CM^2} \qquad (5)$$

From the similitude of the triangles MCD and MAC results : $CM^2=AM.DM$ (6)

From the relations (5) and (6) one infers that

$$m_a\cos\alpha=\frac{a^2}{4}\bullet\frac{1}{DM} \qquad (7)$$

On the other hand, applying the value of the point E to the circle circumscribed to the triangle ABC results that

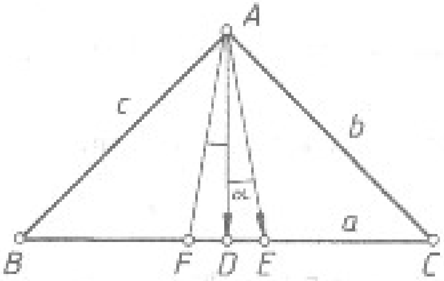
$$m_a EN=\frac{a^2}{4} \qquad (8)$$

From the relations (7) and (8) one obtain that $DM\cos\alpha=EN$, from which one may deduce that the intersection of (DM) with AN is congruous to (EN).

Theorem 4. *If AD is a bisector , AE is a median and AF is a symediane of any triangle ABC and if the lines AD and AF intersect for the second time the circle circumscribed to the triangle ABC in the points M and P, then one gets the* relation :

$$\frac{1}{AD}+\frac{1}{AM}=\frac{2}{AL} \quad where \quad \{L\}=AD\cap EP$$

Proof. From the theorem of cosine and from the theorem of the median results the relation $4m^2_a = b^2+c^2+2bc\cos A$ (1)



On the other hand from the proof of the theorem 1 one deduces that $\frac{m_a}{s_a}=\frac{b^2+c^2}{2bc}$ from which one infers the relation

$$b^2+c^2=\frac{2bcm_a}{s_a} \tag{2}$$

From the relations (1) and (2) results that :

$$4m^2_a=\frac{2bcm_a}{s_a}+2bc\cos A \tag{3}$$

From the relation (3) and from relation $bc=m_a AP$ one infers the relation

$$\frac{2m_a}{AP}=\frac{m_a}{s_a}+\cos A \tag{4}$$

From relation (4) and from the relation $\cos A = 2\cos^2\frac{A}{2}-1$ results that $\frac{2m_a}{AP}=\frac{m_a}{s_a}+2\cos^2\frac{A}{2}-1$ from which one gets the relation below , having in view the fact that:

$$\frac{2m_a}{AP}+2=\frac{m_a}{s_a}+\frac{2p(p-a)}{bc}+1 \tag{5}$$

From the theorem (1) and from the relation (5) one deduces that :

$$\frac{2(m_a+AP)}{AP}=\frac{m_a+s_a}{s_a}+\frac{2m_a i_a\cos\alpha}{bc} \tag{6}$$

As i_a is a bisector in the triangle AEF too, results that $i_a=\frac{2m_a s_a\cos\alpha}{m_a+s_a}$ from which results : $\frac{m_a+s_a}{s_a}=\frac{2m_a\cos\alpha}{i_a}$ (7)

From the relations (6) and (7) one deduces that :

$\frac{m_a + AP}{m_a AP\cos\alpha} = \frac{1}{i_a} + \frac{i_a}{bc}$ from which , having in view the fact that $bc = i_a AM$ results the relation:

$$\frac{m_a + AP}{m_a AP\cos\alpha} = \frac{1}{i_a} + \frac{1}{AM} \qquad (8)$$

As AL is the bisector of the triangle AEP it results that :

$AL = \frac{2m_a AP\cos\alpha}{m_a + AP}$ from which results the relation:

$$\frac{m_a + AP}{m_a AP\cos\alpha} = \frac{2}{AL} \qquad (9)$$

From the relations (8) and (9) one may deduce that :

$$\frac{2}{AL} = \frac{1}{AD} + \frac{1}{AM}, \quad (i_a = AD)$$

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