Buletinul Ştiinţific al Universităţii din Baia Mare
Seria B, Matematică-Informatică, vol. XII (1996), 1- 6

Dedicated to the 35th anniversary of the University of Baia Mare

ABOUT THE STRUCTURE OF THE 2 - RATIONAL GROUPS

ION ARMEANU

The structure of the groups all whose characters are rational valued(the so called rational or Q groups) is relatively well known (see for example [4]). In this note we shall study the structure of the finite groups all of whose irreducible characters are rational valued on the 2 - elements.

The standard notations and terminology are those of [2] for the character theory of groups and [5] for the general theory of groups . All groups will be finite.

Definition . *A 2-rational group is a group all of whose irreducible characters are rational valued on the 2-elements.*

A rational group is a group all of whose irreducible characters are rational valued.

Proposition 1. *A direct product of 2-rational groups is a 2-rational group and a factor group of a 2-rational group is a 2 - rational group.*

Proof. Let G and H be 2-rational groups . Since the irreducible characters of G x H have the form $\alpha\times\beta$ where $\alpha\in Irr(G)$ *and* $\beta\in Irr(H)$ the first part of the statement follows.

The second part follows easily since the irreducible characters of a factor group pulls back to the original group.

The proof of the next proposition is analogous to the proof of the similar proposition for rational groups (see [2]).

Proposition 2. *G is a 2-rational group iff x y in G whenever* $<x>=<y>$ *and x is a 2-element of G.*

By prop. 2. it follows immediately :

Proposition 3. *A group G is a 2-rational group iff*

$$N_G(<x>)/C_G(x)\cong Aut(<x>).$$

Definition . *Let G be a group. An element x of G is 2-central if there exist a Sylow 2-subgroup of G such that* $S\subseteq C_G(x)$.

It is easy to prove:

Proposition 4. *Let G be a 2-rational group . Then:*

i) Every 2 - central 2 - element is an involution .

ii) For every $S\in Syl_2(G)$ *the centre of S, Z(S) is an elementary abelian 2 - group.*

iii) The Sylow 2-subgroup of Z(G) is an elementary abelian 2-group.

Corollary 5. *Let G be a 2-rational group with S an abelian Sylow 2-subgroup . The S is an elementary abelian 2-group.*

Proposition 6. *Let G be a 2-rational group and S a Sylow 2-subgroup of G such that N(<x>) is subnormal in G for every x in S. Then S is a rational group.*

Proof. Let s in S and the group isomorphism

$$f:N(<s>)\to Aut(<s>)$$

Let z,w a set of generators for Aut(<s>) and $x,y \in N(<s>)$ such that f(x)=z and f(y)=w. Since Aut (<s>) is a 2 - group , we can assume that $o(x)=2^i$ and $o(Y)=2^t$. By the subnormality of N(<s>), $S \cap N(<s>)$ is a Sylow 2-subgroup of N(<s>). By Sylow 's theorem there exist u,v in N(<s>) such that $a=x^u$ *and* $b=y^v$ are in S . Then f(a)=f(x)=z and f(b)=f(y)=w and the statement follows.

Proposition 7 . *Let G be a 2-rational group and $S \in Syl_2(G)$ such that for every $P \in Syl_2(G)$ and every noninvolutory $h \in S \cap P$ with $P \cap N(<h>) \in Syl_2(N(<h>))$ and $N(<h>) \cap S \subset P$ there exist $g \in C(h)$ such that $P^g = S$. Then S is a rational group.*

Proof. Analogous to the proof of prop. 6 we obtain $a,b \in P \cap N(<h>)$ such that f(a)= z and f(b)=w.

Definition. *Let H be a permutation group on the set W and let x in H. The cyclic group <x> acts on W. Denote by O(x,w) the orbit of w. We shall say that H is 2r-transversal if for every 2-element x in H, and m an integer relatively prime to o(x), there exist an element h in H such that $x^h = x^m$ and hO(x,w)=O(x,w) for every w in W.*

Using techniques of [3] we shall prove the next two theorems.

Theorem 8. *Suppose GwrH is a 2-rational group. Then both G and H are 2 - rational groups.*

Proof. By the de definition of the wreath product (see[3])H is a factor group of GwrH, hence by prop.1 H is a 2 - rational group.

Let g be in G a 2-element . Define $\pi: W \to G$ by setting $\pi(w)=g$ for every w in W. Then 1*(π)(w)=π(w)=g, therefore 1*(π)=π. Hence o((π;1))=o(g) and (π;1) is an 2-element in GwrH . Then for every positive integer m relatively prime to o(g) there exist (u;h) in GwrH such that $(u;h)(\pi;1)(u;h)^{-1}=(\pi;1)^{*}$. Hence $u\pi_{s}u^{-1}=\pi^{*}$. Since $\pi_{s}=\pi$ it follows that $u(w)gu(w)^{-1}=\pi(w)^{*}=g^{*}$ for every w in W. Hence g is conjugate to g^{*}.

Theorem 9. *Let G be a 2-rational group and (H,W) a 2-r transversal group . Then Gwr(H,W) is a 2-rational group.*

Proof. Let (f;x) in GwrH a 2-element and let m be a positive integer relatively prime to o((f;x)). We have to show that $(f;x)^{r}$ is conjugate to (f;x) . Clearly $(f;x)^{r}=(ff_{x}...f_{x^{r-1}};x^{r})$. . Denote $g=ff_{x}...f_{x^{r-1}}$.

Since H is 2r- transversal , there is an element h in H such that $x^{h}=x^{r}$ and hO(x,w)=O(x,w) for every w in W .

Then $(1;h)(f;x)^{r}(1;h)^{-1}=(g_{h};x)$.

We shall prove now that (g_{h}) is conjugate to (f;x). It is straightforward to prove that $x*(g_{h})(w)=(x*(f))(h^{-1}(w))^{r}$ for every w in W. Then $h^{n}(w)\in O(x,w)$ and therefore $(x^{*}(f)(h^{-1}(w)))^{r}$ is conjugate to x*(f)(w). Hence $x*(g_{h})(w)$ is conjugate to $(x*(f)w))^{r}$ and since G is a 2-r group and $x*(f)(w)\in G$ it follows that $x*(g_{h}(w))$ is conjugate to x*(f)(w).

We shall construct now a map $\mu:W\to G$ such that $(\mu,1)(g_{h},x)(\mu,1)^{-1}=(f,x)$.Let $W=O(x,w_{1})\cup...\cup O(x,w_{q})$ be the pairwise disjoint factors decomposition. Let $|O(x,w_{i})|=s_{i}$. By the previous , there exist $\mu(w_{i})\in G$ such that $\mu(w_{i})x*(g_{h})(w_{i})\mu(w_{i})^{-1}=x*(f)(w_{i})$ *for* $i=1,...,q$.

We define μ on all W by setting

$$\mu(x^{-k}(w_{i}))=\{f(w_{i})...f_{x^{k-1}}(w_{i})\}^{-1}\mu(w_{i})\{g_{h}(w_{i})...g_{h}(x^{-(k-1)}(w_{i})\}$$

for every 1<k<s_{i}-1.

It remains to verify that $(\mu,1)(g_{h},x)(\mu,1)^{-1}=(f,x)$. This follows if we prove that $\mu(w)g_{h}(w)\mu(x^{-1}(w))^{-1}=f(w)$ for every w in W . For $w=w_{i}$ this is obvious . In general , write $w=x^{-k}(w_{i})$ and straightforward follows the statement.

Theorem 10. [1] *Every group G can be embedded in a symmetric group S such that if* $x, y \in G$ *are conjugate in S then* $\langle x \rangle$ *and* $\langle y \rangle$ *are conjugate in G.*

Theorem 11. *A group G can be embedded in a symmetric group S such that the 2-elements of G do not fusion in S iff G is a 2 - rational group.*

Proof. Let G be a 2-rational group embedded in a symmetric group S as in theorem 10. Then x^{q} y in G for some positive integer q. Since x x^{q} in G, the 2-elements of G do not fusion in S.

Reciprocally, let G be embedded in S such that the 2-elements of G do not fusion in S . Let $\chi \in Irr(G)$. For every 2-elemnt x of G we have $\chi^{s}(x) = e\chi(x)$ with e a positive integer. Then $\chi(x)$ is rational and the statement follows.

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Received 1.09.1996

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