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# ON A CLASS OF INTEGRAL FUNCTIONS IN VALUED FIELDS

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**Abstract.** Let P(X) be a polynomial with coefficients in **Z**. We suppose there exists an integral number $N_1$ such that $P(n) > 0$, $\forall\, n \geq N_1$. We consider the set $D = \{1/P(n)\}$, $n \geq N_1$. This paper is devoted to find all the integral functions f(X) with coefficients in **Q** such that $f(D) \subset D$.

Let K be a field admitting a rank 1 nontrivial valuation $|\ |$ (see [1] or [2]). For $x, y \in K$, define $d(x,y) = |x-y|$ and thus (K,d) is a metric space. A formal power series

(1) $$f(X) = \sum_{k=0}^{\infty} a_k X^k \in K[[X]]$$

is called an integral function over K if for every $x \in K$ the sequence

(2) $$S_n(x) = \sum_{k=0}^{n} a_k x^k$$

is a Cauchy sequence. We denote by IK[[X]] the commutative algebra of integral functions over K. If $f(X) \in$ IK[[X]] and $f(K) \subset K$, then

(3) $$Inv_K f = \{D \subset K;\ f(D) \subset D\}$$

defines a topology on K such that K is a locally quasi-compact and locally connected topological space (see [4]).

Suppose K=**Q** and $|\ |$ is the usual absolute value function. Let

(4) $$P(X) = b_s X^s + b_{s-1} X^{s-1} + \ldots + b_1 X + b_0 \in \mathbf{Z}[X],\ b_s > 0,\ s > 0,$$

and we take $N_1 \in N$ such that $P(n) > 0, \forall n \geq N_1$. We consider

(5) $$D = \{y_n = \frac{1}{P(n)}\},\ n \geq N_1$$

and we want to find all the integral functions $f(X) \in$ IQ[[X]] such that $f(D) \subset D$.

**Theorem 1.** *Let $K=Q$, $|\,|$ the usual absolute function and D the set defined by (5). If $f(X)\in IQ[[X]]$ such that $f(D)\subset D$, then $f(X)\in Q[X]$.*

**Proof.** Let $f(X)\in IQ[[X]]$ defined by (1) and $a_i=\alpha_i/\beta_i$, $\alpha_i,\beta_i\in Z$, $\beta_i\neq 0,\forall i\in N$. We suppose $f(D)\subset D$ and $f(X)\notin Q$. Then for every $n\geq N_1$ there exists $k_n\geq N_1$, $k_n\in N$ such that

(6) $$f(\gamma_n)=\gamma_{k_n}=\frac{1}{P(k_n)},\forall n\geq N_1.$$

Since the zeros of an integral function are isolated points (see [6], p.88), for every $\gamma\in D$ there exist only a finite number of elements $x\in D$ such that $f(x)=\gamma$. Now, because f(X) is a continuous functions, it follows that

(7) $$\lim_{n\to\infty}\gamma_{k_n}=0=\lim_{n\to\infty}f(\gamma_n)=f(0)=a_0.$$

Let $x\in[0,c]$, where $c>0$. Since f(X) is an integral function we have

(8) $$\lim_{m\to\infty}\left|a_m^{1/m}\right|=0.$$

Hence

(9) $$|a_m|=\varepsilon_m^m,\ where\ \lim_{m\to\infty}\varepsilon_m=0.$$

We choose $c'>\max\{c+1,1\}$ and then by (9) there exists $M_1>0$ such that

$$\varepsilon_m<\frac{1}{c'},\forall m\geq M_1.$$

Now for every $x\in[0,c]$ and $m\geq M_1$,

$$\left|f(x)-\sum_{j=1}^{m}a_jx^j\right|\leq|a_{m+1}||x|^{m+1}+|a_{m+2}||x|^{m+2}+\ldots\leq\left(\frac{|x|}{c'}\right)^{m+1}+\left(\frac{|x|}{c'}\right)^{m+2}+\ldots=\frac{\left(\frac{|x|}{c'}\right)^{m+1}}{1-\frac{|x|}{c'}}=$$

$$\frac{|x|^{m+1}}{c'^m\,(c'-|x|)}<|x|^{m+1},\forall x\in[0,c],\forall m\geq M_1.$$

Hence

(10) $$\left|f(x)-\sum_{j=1}^{m}a_jx^j\right|<|x|^{m+1},\forall x\in[0,c],\forall m\geq M_1.$$

Let $a_i$ be the first coefficient such that $a_i\neq 0$. Then $i\geq 1$. We denote by

(11) $$M_2=\max\{M_1,2i\}.$$

If f(X) is not a polynomial we can suppose that there exists a fixed $m\geq M_2$ such that $a_m\neq 0$. By (10) it follows that

$$\left|\frac{1}{P(k_n)}-\sum_{j=1}^{m}\frac{\alpha_j}{\beta_j}\cdot\frac{1}{P^j(n)}\right|<\frac{1}{P^{m+1}(n)},\forall n\geq N_1.$$

Hence

$$\lim_{n\to\infty}\frac{P^i(n)}{P(k_n)}=\frac{\alpha_i}{\beta_i} \tag{12}$$

and

$$\left|\beta_m P^m(n)-P(k_n)[\alpha_i\beta_m\beta_i^{-1}P^{m-i}(n)+\alpha_{i+1}\beta_m\beta_{i+1}^{-1}P^{m-i-1}(n)+\ldots+\alpha_m]\right|<\frac{\beta_m P(k_n)}{P(n)}. \tag{13}$$

We denote by

$$r_m(n)=\beta_m P^m(n)-P(k_n)[\alpha_i\beta_m\beta_i^{-1}P^{m-i}(n)+\ldots+\alpha_m]\in\mathbf{Q},\forall n\geq N_1. \tag{14}$$

It now follows from (12) and (13) that

$$\left|\frac{r_m(n)}{P^{i-1}(n)}\right|\leq\frac{\beta_m P(k_n)}{P^i(n)}$$

and

$$r_m(n)=O(P^{i-1}(n)). \tag{15}$$

We now consider the polynomials

$$P_1(X)=\beta_m P^m(X)\in\mathbf{Z}[X],$$

$$P_2(X)=\alpha_i\beta_m\beta_i^{-1}P^{m-i}(X)+\ldots+\alpha_m\in\mathbf{Q}[X].$$

Then there exist $Q_1(X)$, $R_1(X)\in\mathbf{Q}[X]$ such that

$$P_1(X)=Q_1(X)P_2(X)+R_1(X), \tag{16}$$

where $\deg R_1(X)<(m-i)\deg P(X)$, $\deg Q_1(X)=i\deg P(X)$, $Q_1(X)=Q_2(P(X))$, with $Q_2(X)\in\mathbf{Q}[X]$ and $\deg Q_2(X)=i$.

Using (14) and (16) we may write

$$P(k_n)=Q_1(n)+\frac{R_1(n)-r_m(n)}{P_2(n)}. \tag{17}$$

Since $m>2i$, by (15), it follows that

$$\lim_{n\to\infty}\frac{R_1(n)-r_m(n)}{P_2(n)}=0.$$

Hence we can find $N_2>N_1$ such that for every $n\geq N_2$ we have

$$\left|\frac{R_1(n)-r_m(n)}{P_2(n)}\right|<\frac{1}{d+1}, \tag{18}$$

where d is the least common multiple of the denominator of the coefficients of $Q_1(X)$. Because $P(k_n)\in N$, by (17) it follows that

$$P(k_n)=Q_1(n),\forall n\geq N_2.$$

Hence

(19) $$f\left(\frac{1}{P(n)}\right)=\frac{1}{Q_1(n)},\ \forall n\geq N_2.$$

Since $Q_1(X)=Q_2(P(X))$, by (19), we have

(20) $$f\left(\frac{1}{P(n)}\right)=\frac{P^{-i}(n)}{Q_3(P^{-1}(n))},\ \forall n\geq N_2,$$

where

$$Q_3(X)=P^i(X)Q_2(\frac{1}{P(X)}).$$

But the analytic functions $f(X)$ and $g(X)=X^i/Q_3(X)$ have the same value at an infinity of points which have an accumulation point and then they must be identically equal. On the other hand $f(X)$ is an integral function and we obtain $Q_3(X)=q\in\mathbf{Q}$. So $f(X)=X^i/q$, which gives a contradiction, if $m>2i$. This proves the theorem. □

**Lemma 1.** *Let $f(X)\in Q[X]$ with $f(0)=0$ and let D be the set defined by (5). If $f(D)\subset D$, then*

(21) $$f(X)=\frac{\alpha_i}{\beta_i}\cdot X^i,\ \alpha_i,\beta_i\in\mathbf{Z}^*,\ i\geq 1.$$

**Proof.** Let us write

(22) $$f(X)=\frac{\alpha_i}{\beta_i}\cdot X^i+\frac{\alpha_{i+1}}{\beta_{i+1}}\cdot X^{i+1}+\ldots+\frac{\alpha_m}{\beta_m}X^m,$$

where $\alpha_j,\beta_j\in\mathbf{Z}$, $(\alpha_j,\beta_j)=1$, $\forall j=i,i+1,\ldots,m$. Since

$$f\left(\frac{1}{P(n)}\right)=\frac{1}{P(k_n)},\ \forall n\geq N_1,$$

where $P(X)$ is the polynomial defined by (4), it follows that

$$\frac{\alpha_i}{\beta_i P^i(n)}+\frac{\alpha_{i+1}}{\beta_{i+1}P^{i+1}(n)}+\ldots+\frac{\alpha_m}{\beta_m P^m(n)}=\frac{1}{P(k_n)},\forall n\geq N_1.$$

Hence

(23) $$P(k_n)=\frac{\beta_i\beta_{i+1}\ldots\beta_m P^m(n)}{S(n)},$$

where

(24) $$S(X)=\alpha_i\beta_{i+1}\ldots\beta_m P^{m-i}(X)+\alpha_{i+1}\beta_i\beta_{i+2}\ldots\beta_m P^{m-i-1}(X)+\ldots+\alpha_m\beta_i\ldots\beta_{m-1}\in\mathbf{Z}[X].$$

We may now write

(25) $$P(k_n)=\frac{\beta_i}{\alpha_i}\cdot P^i(n)+\theta_{i-1}P^{i-1}(n)+\ldots+\theta_0+\frac{R_1(n)}{S(n)},$$

where $\theta_j\in\mathbf{Q}$, $j=0,1,\ldots,i-1$, $R_1(X)\in\mathbf{Q}[X]$ and $\deg R_1(X)<\deg S(X)$.

But $R_1(X)=0$. For

$$\lim_{n\to\infty}\frac{R_1(n)}{S(n)}=0,$$

$P(k_n)\in N$ and we can choose $N_2\in N$ such that

$$\left|\frac{R_1(n)}{S(n)}\right|<\frac{1}{d+1},\forall n\geq N_2,$$

where d is the least common multiple of the denominators of the coefficients of the polynomial

$$R_2(X)=\frac{\beta_i}{\alpha_i}\cdot P^i(X)+\theta_{i-1}P^{i-1}(X)+\ldots+\theta_0\in\mathbf{Q}[X].$$

Hence

$$P(k_n)=\frac{\alpha_i}{\beta_i}\cdot P^i(n)+\theta_{i-1}P^{i-1}(n)+\ldots+\theta_0 \tag{26}$$

and

$$\beta_i\beta_{i+1}\ldots\beta_m P^m(X)=S(X)R_2(X). \tag{27}$$

Then all the roots of S(X) are also roots of P(X) and $\alpha_m=0$. So

$$f(X)=\frac{\alpha_i}{\beta_i}\cdot X^i \tag{28}$$

and the result follows. □

**Lemma 2.** *Let $f(X)=X^i/\gamma\in Q[X]$, $\gamma\in Q,\gamma>0$, $i\geq 1$, such that $f(D)\subset D$. Then there exist $Q(X)\in Q[X]$ and $N_2\in N$ such that*

$$k_n=Q(n),\forall n\geq N_2, \tag{29}$$

*where $k_n$ is defined by (6).*

**Proof.** We consider the algebraic function $y=y(x)$ defined by $P(y)=\gamma P^i(x)$ or

$$b_s y^s+b_{s-1}y^{s-1}+\ldots+b_1y+b_0=\gamma(b_s x^s+b_{s-1}x^{s-1}+\ldots+b_0)^i. \tag{30}$$

We put $z=1/x$. Then by (30) we have

$$z^{is}(b_s y^s+b_{s-1}y^{s-1}+\ldots+b_1y+b_0)=\gamma(b_s+b_{s-1}z+\ldots+b_0z^s)^i.$$

We denote by $u=z^iy$. Now it follows that

$$b_s u^s+z^ib_{s-1}u^{s-1}+z^{2i}b_{s-2}u^{s-2}+\ldots+z^{si}b_0-\gamma(b_s+b_{s-1}z+\ldots+b_1z^{s-1}+b_0z^s)^i=0. \tag{31}$$

Using Puiseux's Theorem (see [3], p.118 or [5]) we can write the algebraic functions defined by (31) using power series with coefficients in C. So we obtain a solution

$$u(z)=c_0+c_1z+c_2z^2+\ldots, \tag{32}$$

where $c_0=u(0)=(\gamma b_s^{i-1})^{1/s}$ and all the solutions have the first coefficient in (32) of the form $c_0\zeta^r$, where $r\in N$ and $\zeta$ is a primitive s- roots of unity. Hence we obtain the single solution of (30)

(33) $$y(x)=\sqrt[i]{\gamma b_s^{i-1}}x^i+c_1x^{i-1}+\ldots+c_{i-1}x+c_i+\sum_{j=1}^{\infty}d_jx^{-j},$$

such that $c_0\in\mathbf{R}$, $c_0>0$. Here the series $\sum_{j=1}^{\infty}d_jz^j$ has radii of convergence $\neq 0$.

Since $f(D)\subset D$ we have

$$P(k_n)=\gamma P^i(n).$$

Hence if follows easily that

(34) $$k_n=\sqrt[i]{\gamma b_s^{i-1}}n^i+c_1n^{i-1}+\ldots+c_{i-1}n+c_i+O\left(\frac{1}{n}\right),\forall n\geq N_1.$$

Now we compute the finite difference of i-th order of $k_n$

$$\delta_n=\Delta^ik_n=k_{n+i}-C_n^1k_{n+i-1}+C_n^2k_{n+i-2}-\ldots+(-1)^ik_n=i!\sqrt[i]{\gamma b_s^{i-1}}+O\left(\frac{1}{n}\right),\forall n\geq N_1.$$

Since $\delta_n\in\mathbf{Z},\forall n\geq N_1$ it follows that

(35) $$i!\sqrt[i]{\gamma b_s^{i-1}}=\tau\in\mathbf{Z}$$

Using the Theorem of Implicit Function it follows now from (32) that $c_j\in\mathbf{Q}$,j=1,2,…,i and then there exists $N_2\in\mathbf{N}$, $N_2>N_1$ such that

(36) $$k_n=\frac{\tau}{i!}n^i+c_1n^{i-1}+\ldots+c_{i-1}n+c_i.$$

Hence $k_n=Q(n)$, where

(37) $$Q(X)=\frac{\tau}{i!}X^i+c_1X^{i-1}+\ldots+c_{i-1}X+c_0\in\mathbf{Q}[X]$$

and the result follows. □

**Lemma 3.** *Let P(X) defined by (4) and $Q(X)\in\mathbf{Q}[X]\setminus\{X\}$ such that $\exists N_2\in\mathbf{N}$ and $Q(x)\geq 0$, $\forall x\geq N_2$. We consider $\sigma\in\mathbf{Q}^*$ and suppose*

(38) $$b_s^{i-1}P(Q(X))=\sigma^sP^i(X),\ i\geq 1.$$

*Then we have*

(39) $$P(X)=b_s(X+\frac{b_{s-1}}{sb_s})^s$$

*and*

(40) $$Q(X)=\sigma(X+\frac{b_{s-1}}{sb_s})^i-\frac{b_{s-1}}{sb_s},$$

**Proof.** Let

$$P(X)=b_s(X-x_1)^{r_1}\ldots(X-x_t)^{r_t},$$

where $x_1,\ldots,x_t$ are the distinct roots of P(X). Then by (38) it follows that

$$b_s^i(Q(X)-x_1)^{r_1}(Q(X)-x_2)^{r_2}\dots(Q(X)-x_t)^{r_t})=\sigma^s b_s^i(X-x_1)^{r_1 i}\dots(X-x_t)^{r_t i}.$$

Hence the roots of the polynomial P(Q(X)) are $x_1,\dots,x_t$ and every factor $\left(Q(X)-x_j\right)^{r_j}$ has only one root $x_{k(j)}$ with the multiplicity $r_{k(j)}i$. So

$$Q(X)-x_j=\sigma(X-x_{k(j)})^i, \tag{41}$$

where j=1,2,…,t and k(1),…,k(t) is a permutation of 1,…,t. By (41) it follows that Q'(X) has t(i-1) roots which gives a contradiction, if i>1 and t>1, because deg Q'(X) = i-1.

If i=1 and t>1, by (41) it follows that $Q(x_k)=x_k$ for at least two values of k or we have r distinct roots $x_{i_1},\dots,x_{i_r}$, where r≥2 such that $Q(x_{i_1})=x_{i_2},\dots,Q(x_{i_r})=x_{i_1}$. Since deg Q(X) = 1 it follows easily that σ=±1. Hence Q(X)=X and this proves the lemma. □

**Theorem 2.** *Let D be the set defined by (5). Then there exists $f(X)\in IQ[[X]]\setminus\{X\}$ such that $f(D)\subset D$ if and only if*

$$f(X)=\frac{1}{\gamma_i}X^i,\ \gamma_i\in\mathbf{Q}_+^*,\ i\ge 1,$$

*P(X) is given by (39) and if Q(X) is the polynomial given by (40), where*

$$\sigma=\sigma_i=\sqrt[s]{b_s^{i-1}\gamma_i},$$

*then the finite differences $\Delta^j Q(0)\in Z$, $\forall j=0,1,\dots,i$.*

**Proof.** If f(D) ⊂ D, by Theorem 1 and Lemma 1 it follows that

$$f(X)=\frac{1}{\gamma_i}X^i,\ \textit{where}\ \gamma_i=\frac{\beta_i}{\alpha_i}\in\mathbf{Q}.$$

Using Lemma 2 and Lemma 3 with $\sigma=\sigma_i$ we obtain $k_n=Q(n)$, P(X) is given by (39) and Q(X) is given by (40). Since $Q(n)\in N$, $\forall n\ge N_2$, it follows that $Q(Z)\subset Z$ and using Newton's interpolation polynomial we have $\Delta^j Q(0)\in\mathbf{Z}$, $\forall j=0,1,\dots,i$.

The converse is obvious. □

**Example.** Let $P(X)=2(3x+1)^2$, $N_1=0$. Then

$$\sigma_i=\sqrt{18^{i-1}\gamma_i}\ \textit{and}\ Q(X)=3^{i-1}\sqrt{2^{i-1}\gamma_i}\,(X+\frac{1}{3})^i-\frac{1}{3}.$$

If f(D)⊂D, since $\Delta^0 Q(0)=Q(0)=\dfrac{\sqrt{2^{i-1}\gamma_i}}{3}-\dfrac{1}{3}\in\mathbf{Z}$, it follows that

$$\sqrt{2^{i-1}\gamma_i}=3k+1, k\in\mathbf{N}\ \textit{and}\ f(X)=\frac{2^{i-1}}{(3k+1)^2}X^i, k\in\mathbf{N}, i\ge 1.$$

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