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THE COMPLETION OF A GAUSS TYPE VALUATION FIELD

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1.Let (K, v) be a *local field*, K being a commutative field and v a discrete and rank one valuation on K, such that K is complete with respect to v. For a $\delta \in \mathbf{Q}$, one denotes by $P_\delta(K, v)$ the set of all Laurent series $\alpha = \sum_{n\in \mathbf{Z}} a_n X^n$, $a_n \in \mathbf{K}$,for every $n \in \mathbf{Z}$, and

a) there exist $M \in \mathbf{R}$ with $v(a_n) + n\delta \geq M$, for every $n \in \mathbf{Z}$,

b) $\lim_{n\to -\infty}(v(a_n) + n\delta) = \infty$.

If we define $u_\delta(\alpha) = \inf_{n\in\mathbf{Z}}(v(a_n) + n\delta)$, u_δ is a rank one and discrete valuation on $P_\delta(K, v)$ and this one is a local field with respect to u_δ . We call $P_\delta(K, v)$ a *Parshin field* ([4]).

In [1] we find a description of the completion of the rational function field $K(X)$ with respect to the so called Gauss valuation u_0 , where $u_0(a_0 + a_1X + ... + a_nX^n) = \inf_n(v(a_n))$,as a subfield in the Parshin field $P_0(K, v)$.Now, if we introduce on $K(X)$ a Gauss valuation u_δ,where $u_\delta(a_0 + a_1X + ... + a_nX^n) = \inf_n(v(a_n) + n\delta)$, it is not difficult to do the same type of reasoning as in [1] in order to describe the completion of $K(X)$ with respect to u_δ in the Parshin field $P_\delta(K, v)$.

2.Let a be an algebraic element over K and $K' = K(a)$ the corresponding algebraic extension of the field K.For a $\delta \in \mathbf{Q}$,we denote by $P_\delta(K, v, a)$ the set of all Laurent series $\alpha = \sum_{n\in\mathbf{Z}} a_n(X - a)^n$, $a_n \in K'$ with the following properties:

c) there exist $M \in \mathbf{R}$ with $v'(a_n) + n\delta \geq M$, for every $n \in \mathbf{Z}$

d)$\lim_{n\to -\infty}(v'(a_n) + n\delta) = \infty$,where $v'$is the unique extension of the valuation v to K'.

If we define $u_\delta = \inf_{n\in\mathbf{Z}}(v'(a_n) + n\delta)$, $P_\delta(K, v, a)$ becomes a local field with respect to u_δ.

3.If $P(X) = a_0 + a_1X + ... + a_nX^n$ is a polynomial in $K[X]$,we consider $P(X) = a'_0 + a'_1(X - a) + ... + a'_n(X - a)^n$, the Taylor's expansion of $P(X)$ in the element $a \in K'$.

Let us define $v_\delta(P(X)) = \inf_n(v'(a'_n) + n\delta)$. It is not difficult to prove that v_δ is a valuation on $K(X)$. The problem is how to describe the completion of $K(X)$ with respect to v_δ. It is clear enough that v_δ is a Gauss type valuation on $K'(X)$, and the completion of $K'(X)$ with respect to this last valuation is possible to describe as a subfield in $P_\delta(K, v, a)$([1]). Using Taylor's expansions in a, it is possible to construct an embedding of $K(X)$ in $K'(X)$ and then in $P_\delta(K, v, a)$.So the completion of $K(X)$

with respect to v_δ is exactly the topological closure of $K(X)$ in $P_\delta(K, v, a)$. It remains only to say when a polynomial from $K'(X)$ is a Taylor's expansion of a polynomial from $K(X)$.

Theorem 1

A polynomial $P^*(X) = a'_0 + a'_1(X-a) + ... + a'_n(X-a)^n$is the Taylor extension of $P(X) = a_0 + a_1X + ... + a_nX^n$ if and only if we have the following matrix equality :

$\mathbf{A_n} \times (a'_0, a'_1, ..., a'_n)^t = (a_0, a_1, ..., a_n)^t$,where $\mathbf{A_n} = (a_{ij}),\ i, j \in \{0, 1, ..., n\}$,with $a_{ij} = 0$ if $i > j$, $a_{ij} = 1$ if $i = j$and $a_{ij} = C^i_{j-i}\, a^{j-i}$, if $i < j$.($C^m_n = \frac{n!}{m!(n-m)!}$).The matrix $\mathbf{A_n}$ is invertible in $\mathbf{Z}[a]$ and the image of $K(X)$ in $K'(X-a)$ is $\left\{ \frac{a'_0 + a'_1(X-a) + ... + a'_n(X-a)^n}{b'_0 + b'_1(X-a) + ... + b'_m(X-a)^m} \right\}$, where $(a'_0, a'_1, ..., a'_n) = \mathbf{A_n}^{-1} \times (a_0, a_1, ..., a_n)$ and $(b'_0, b'_1, ..., b'_m) = \mathbf{A_m}^{-1} \times (b_0, b_1, ..., b_m)$, with $a_i, b_j \in K$, $i = \overline{0, n}$, $j = \overline{0, m}$.

Corollary 2 $P^*(X)$ is an element from $K[X]$,if and only if $\mathbf{A_n}(a'_0, a'_1, ..., a'_n)^t$ is a vector in K^n.

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