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ON A GENERALIZED DURRMEYER OPERATORS

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1.

Durrmeyer [5] defined a new kind of modified Bernstein polynomial operators on $L_1[0,1]$, the space of Lebesgue integrable functions on [0,1], as :

$$(M_n f)(x) = (n+1)\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_0^1 \binom{n}{k} t^k (1-t)^{n-k} f(t)\, dt \quad (1)$$

The aim of this note is to present a general class of linear positive operators $(L_n)_{n\geq 1}$ of integral type. This construction contains as particular cases well-known operators introduced and studied during the time by many authors. We evaluate the order of approximation in terms of the moduli of smoothness ω, ω_2, and indicate sufficient conditions which ensure the uniform convergence of the sequence. In the last section of this paper we apply our result to operators which represent a generalization of Stancu's operators. We mention that our estimation improves a previous result.

2.

Let J and D be subintervals of real axis and for each integer $n \geq 1$ let I_n a set of indexes. We consider p_{nk} , q_{nk} real non - negative valued functions with the properties $p_{nk} \in \mathbf{C}(J)$, $q_{nk} \in \mathbf{C}(J \times D)$, and for any $x \in J$ $q_{nk}(x , \cdot)$ is bounded on D, where $n \geq 1$ and $k \in I_n$. The operators in question are defined on the space

$$L_1(D) = \{f: D \to \mathbf{R} \,/\, \int_D |f(x)|\, dx < \infty \}$$

by

$$(L_n f)(x) = \sum_{k \in I_n} p_{nk}(x) \int_D q_{nk}(x,y)\, f(y)\, dy \qquad (2)$$

It is obvious that this operator is linear and positive. Now we shall analyse three particular cases.

A. Firstly, we choose $D = J = [0,1]$ and $I_n = \{0, 1, 2, \dots , n\}$. If we put

$$p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k} \qquad \text{and} \quad q_{nk}(x,y) = (n+1)\, p_{nk}(y)$$

we obtain the operator defined at (1). M. M. Derriennic [4] studied these modified Bernstein polynomials. He demonstrated that if the derivative $d^r f / dx^r$ with $r \geq 0$ is continuous on $[0,1]$ then $(d^r/dx^r) M_n f$ converges uniformly on $[0,1]$ and $\sup_{x \in [0,1]} |(M_n f)(x) - f(x)| \leq 2\omega(n^{-1/2})$ where ω is the modulus of continuity of f. He has also proved that if f belongs to Sobolev space $W^{l,p}(0,1)$ with $l \geq 0$, $p \geq 1$, then $M_n f$ converges to f in $W^{l,p}(0,1)$.

B. Secondly, we choose $D = J = [0, \infty)$ and $I_n = \mathbf{N}$. Now we take

$$p_{nk}(x) = \binom{n+k-1}{n} x^k (1+x)^{-n-k} \quad , \qquad q_{nk}(x,y) = (n-1)\, p_{nk}(y)$$

then L_n become operators proposed by A. Sahai and G. Prasad [7] and termed as modified Lupaş operators. The authors established theorems concerning asymptotic approximation and error estimation in the simultaneous approximation by these operators. Later, R.P. Sinha, T.A.K. Sinha, P.N. Agrawal and V. Gupta [8], [9] enlarged the study upon L_n. Using the device of Steklov means they obtained the estimate of error in L_p - approximation in terms of higher order integral modulus of smoothness. We note that the above operators also should be called modified Baskakov operators because Baskakov [2] has introduced the operators :

$$(K_n f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right).$$

If we replace $f(\frac{k}{n})$ by $(n-1)\int_0^{\infty} p_{nk}(t)\, f(t)\, dt$, we obtain the operators

discussed at this example.

C. Finally, we choose $D = J = [0,1]$ and $I_n = \{ 0, 1, 2, \dots , n-r \}$ where r is an integer parameter such that $2r < n$. We define :

$$p_{nk}(x) = \binom{n-r}{k} x^k (1-x)^{n-r-k}$$

and

$$q_{nk}(x,y) = (1-x)\, A_{nrk}\, (1-y)\, p_{nk}(y) + x\, B_{nrk}\, y\, p_{nk}(y) \qquad (3)$$

where the constants A_{nrk} , B_{nrk} have the following form :

$$A_{nrk} = \frac{(n-r+1)(n-r+2)}{n-r-k+1}, \quad B_{nrk} = \frac{(n-r+1)(n-r+2)}{k+1}, \quad 0 \le k \le n-r.$$

This operator was studied by Chen Wenzhong and Tian Jishan [3]. They computed the degrees of approximations for continuous functions and L_p - functions. In this case L_n represents a generalization of an operator introduced by D.D. Stancu [10]. The expression of Stancu's operator is the following:

$$(L_{m,r}^{\alpha,\beta}f)(x) = \sum_{k=0}^{m-r} \binom{m-r}{k} x^k (1-x)^{m-r-k} \left[(1-x) f\left(\frac{k+\alpha}{m+\beta}\right) + x f\left(\frac{k+r+\alpha}{m+\beta}\right) \right]$$

where r is a non - negative integer ($2r < n$) and α, β are real parameters so that $0 \leq \alpha \leq \beta$.

$L_{m,r}^{\alpha,\beta}$ was introduced using a probabilistic method. A special attention was given to the case of the operator $L_{m,r} = L_{m,r}^{0,0}$. D.D. Stancu proved that the remainder of the approximation formula of a function $f \in \mathbf{C}[0,1]$ by $L_{m,r}f$ can be represented either by means of divided differences, or by an integral form, obtained by using a classical theorem of Peano. He also showed that $L_{m,r}$ admits the variation diminishing property, he determined the point spectrum of $L_{m,r}$ and constructed a quadrature formula.

3.

In this section we are concerned with the estimate of the order of approximation of a function f by means of the linear operator L_n.

<u>**Theorem A.**</u> Let be the sequence defined by (2) and $K \subset J$ a compact with the property that f is continuous on K. For every $x \in K$ let be

$$\int_D q_{nk}(x,y)\,(x-y)^r\,dy = \mu_{nk}(r\,;x) \quad , \qquad r \in \{0,2\} \tag{4}$$

and

$$\sum_{k \in I_x} p_{nk}(x)\,\mu_{nk}(0\,;x) = a_n(x) \tag{5}$$

If i) two real numbers A, B exist so that $0 < A \leq a_n(x) \leq B$, $x \in K$

ii) $\alpha > 0$ exists so that $\sum_{k \in I_x} p_{nk}(x)\,\mu_{nk}(0\,;x)\,\mu_{nk}(2\,;x) = 0(n^{-\alpha})$

then the following inequality

$$| (L_n f)(x) - f(x) | \leq B \gamma_n |f(x)| + (B + 0(n^{-\alpha + 1/2})) \omega (f, n^{-1/2}) \quad (6)$$

holds, where $\gamma_n = \sup_{x \in K} | 1 - a_n^{-1}(x) |$.

Proof By using (5) we get :

$$f(x) = \frac{f(x)}{a_n(x)} \sum_{k \in I_n} p_{nk}(x) \int_D q_{nk}(x,y)\, dy$$

Taking into account the above relation and (2) we deduce :

$$| (L_n f)(x) - f(x) | \leq \sum_{k \in I_n} p_{nk}(x) \int_D q_{nk}(x,y) \left| f(y) - \frac{f(x)}{a_n(x)} \right| dy \quad (7)$$

We shall use the modulus of continuity ω defined by :

$$\omega(\delta) = \omega(f,\delta) = \sup | f(x'') - f(x') | ,$$

where x' and x'' are points from K so that $| x'' - x' | < \delta$, δ being a positive number. Using the following well-known properties of ω

$$| f(x'') - f(x') | \leq \omega (| x'' - x' |) \leq (1 + \delta^{-1} | x'' - x' |) \omega(f,\delta)$$

and the definition of γ_n we can write successively :

$$\left| f(y) - \frac{f(x)}{a_n(x)} \right| \leq | f(y) - f(x) | + | f(x) | \left| 1 - \frac{1}{a_n(x)} \right| \leq$$

$$\leq (1 + \delta^{-1} |x - y|) \omega(f,\delta) + \gamma_n | f(x) | \quad (8)$$

Substituting (8) in relation (7) we obtain :

$$| (L_n f)(x) - f(x) | \leq$$

$$\leq \gamma_n | f(x) | a_n(x) + (a_n(x) + \delta^{-1} \sum_{k \in I_n} p_{nk}(x) \int_D q_{nk}(x,y) |x - y| dy) \omega(f,\delta) \quad (9)$$

Now, we apply the Cauchy inequality :

$$\int_D q_{nk}(x,y) |x - y| dy \leq \int_D q_{nk}(x,y) dy \int_D q_{nk}(x,y) (x - y)^2 dy =$$

$$= \mu_{nk}(0 ; x) \mu_{nk}(2 ; x) \quad (10)$$

If we insert $\delta = n^{-1/2}$, the relations (9) , (10) and the hypotheses of this theorem lead us to the desired result.

The modulus of smoothness ω_2 is also frequently used in quantitative approximation. I. Gavrea and I. Raşa [6] established the inequality :

$$\omega(f,\delta) \le \left(3 + \frac{l(k)}{\delta}\right) \omega_2(f,\delta) + \frac{6\delta}{l(k)} \| f \| , \qquad 0 < \delta \le l(k)$$

where l(k) represents the length of the interval compact K and $\|\cdot\|$ is sup-norm on **C**(K).

We mention that :

$\omega_2(f,\delta) = \sup \{ | f(x + 2t) - 2f(x+t) + f(x) | : 0 \le t \le \delta , x \in K , x + 2t \in K \}$

After a few computations we can state :

Theorem B If the notations and conditions required by theorem A work then the following inequality

$$| (L_n f)(x) - f(x) | \le B (\gamma_n + 6l^{-1}(k)n^{-1/2} + 0(n^{-\alpha}) \| f \| +$$
$$+ B(3 + l(k)n^{1/2} + 0(n^{-\alpha+1}))\, \omega_2(f, n^{-1/2})$$

holds.

Once inequality (6) is known, we can easily obtain :

COROLLARY Under the conditions of theorem A, if $\alpha > 1/2$ and γ_n converges uniformly to zero, then

$$\lim_{n \to \infty} (L_n f)(x) = f(x) \quad ,$$

the convergence being uniform on K.

4.

We come back at example **C**. It is known that the Beta function is defined as:

$$B(l,m) = \int_0^1 x^{l-1} (1-x)^{m-1} dx = \frac{\Gamma(l)\, \Gamma(m)}{\Gamma(l+m)}$$

and $\Gamma(s) = (s-1)!$ for $s \ge 1$ integer.

For q_{nk} defined in (3), this formula helps us to prove that

$$\int_0^1 q_{nk}(x,y)\,dy = 1 = \mu_{nk}(0\,;x)$$

$$\int_0^1 y\, q_{nk}(x,y)\,dy = \frac{x+k+1}{n-r+3}$$

$$\int_0^1 y^2\, q_{nk}(x,y)\,dy = \frac{2(k+2)x+(k+1)(k+2)}{(n-r+3)(n-r+4)}$$

Consequently, we get :

$$\mu_{nk}(2\,;x) = \frac{n-r+1}{n-r+3}x^2 - 2\,\frac{(n-r+3)k+(n-r+2)}{(n-r+3)(n-r+4)}x + \frac{(k+1)(k+2)}{(n-r+3)(n-r+4)}.$$

In this case we have $a_n(x) = 1$, $A = B = 1$, $\gamma_n = 0$.

By using the identities

$$\sum_{k=1}^{n-r} k\binom{n-r}{k}x^k(1-x)^{n-r-k} = (n-r)x$$

$$\sum_{k=2}^{n-r} k(k-1)\binom{n-r}{k}x^k(1-x)^{n-r-k} = (n-r)(n-r-1)x^2,$$

we obtain

$$\sum_{k=0}^{n-r} p_{nk}(x)\,\mu_{nk}(0\,;x)\,\mu_{nk}(2\,;x) =$$

$$= \frac{2(n-r-2)}{(n-r+3)(n-r+4)}\left(x - x^2 + \frac{1}{n-r-2}\right) \le \frac{1}{2(n-r+5)}$$

because $x - x^2 \le 1/4$, $x \in [0,1]$.

According to theorem A, all these relations lead us to the following inequality :

$$|(L_nf)(x) - f(x)| \le \left(1 + \frac{\sqrt{n}}{2(n-r+5)}\right)\omega(f, n^{-1/2}).$$

Our result improves the one given in [3] because there the coefficient of ω was found as being 2.

We mention that in the paper [1] was presented an extension of operator $L_{m,r}^{\alpha,\beta}$ in the sense of Kantorovich.

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REFERENCES

1. **AGRATINI,O.** *Approximation properties of a class of operators of Stancu - Kantorovich type*, Research Seminar on Numerical and Statistical Calculus, "Babeş-Bolyai" Univ., Cluj-Napoca, Preprint **1** (1994), 3 -12 .
2. **BASKAKOV, V.A.** *An example of linear positive operators in the space of continuous functions*, Dokl. Akad. Nauk. SSSR, **113** (1957), 249 - 251 .
3. **CHEN WENZHONG, TIAN JISHAN** *On approximation properties of Stancu operators of integral type*, Journal of Xiamen University, **26** (1987), 3, 270 - 276 .
4. **DERRIENNIC, M.M.** *Sur l'approximation de fonctions intégrables sur [0,1] par des polynômes de Bernstein modifies*, Journal Approx. Theory, **31** (1981), 325 - 343 .
5. **DURRMEYER, J.L.** *Une formule d'inversion de la transformée de Laplace : Application à la thèorie des moments*, Thèse de 3e cycle, Faculté des Sciences de l'Université de Paris, 1967 .

6. **GAVREA, I., RAŞA, I.** *Remarks on some quantitative Korovkin - type results*, Revue d'Analyse Numerique et de Theorie de l'Approximation, **22** (1993), 2, 173 - 176 .

7. **SAHAI, A., PRASAD, G.** *On simultaneous approximation by modified Lupas operators*, Journal Approx. Theory, **45** (1985), 122 - 128 .

8. **SINHA, T.A.K., AGRAWAL, P.N., GUPTA, V.** *On simultaneous approximation by modified Baskakov operators*, Bull. Soc. Math. Belg. **43** (1991), 2, Ser. B, 217 - 231.

9. **SINHA, T.A.K., AGRAWAL, P.N., SAHAI, A., GUPTA, V.** *Improved L_p - approximation by linear combination of modified Lupas operators*, Bull. Inst. Math. Academia Sinica, **17** (1983), 3, 243 - 253 .

10. **STANCU, D.D.** *Approximation of functions by means of a new generalized Bernstein operator*, Calcolo, **20** (1983), 2, 211 - 229 .

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